Math 262: Calculus II
Introduction to Sequences

Definition: A **sequence** is a function f whose domain is the set of positive integers. We will almost always take its range to be a subset of the real numbers.

Example: 1, 2, 4, 8, 16, ...

Notation: Since any sequence is a function $f : \mathbb{N} \to \mathbb{R}$, the *terms* of a sequence (the outputs of f) are in one-to-one correspondence with the positive real numbers, so we write them in order as: $a_1, a_2, a_3, ...a_n, ...$, where $a_n = f(n)$ is called the n**th term** of the sequence.

When the terms of a sequence are given by a function $f(n)$ with an explicit formula, we often abbreviate the sequence by writing it as either $\{f(n)\}_{n=1}^{\infty}$ or just $\{f(n)\}$.

Sometimes, $f(n)$ is *not* given explicitly. Instead, we may only be given a way of computing new terms in the sequence in terms of previous terms. Such sequences are said to be **recursively defined**. One example of such a sequence is the *Fibonacci Sequence* given by $a_1 = 1$, $a_2 = 1$ and for $n \geq 3$, $a_n = a_{n-1} + a_{n-2}$.

Write out the first 10 terms of the Fibonacci sequence:

Claim: In our example above, we can think of the sequence as having been given by the rule $f(n) = 2^{n-1}$.

The Graph of a sequence:

Example:

More Examples: Write out the first 5 terms of each of the following sequences.

1. $\{2n - 1\}$

2. $\left\{(-1)^n \frac{1}{n}\right\}$

3. $\{3\}$

The Limit of a Sequence

Example: Consider the sequence $\left\{1 + \frac{1}{n}\right\}$. What happens to the terms of this sequence as $n \to \infty$?

Definition: A sequence $\{a_n\}$ has **limit L** or **converges to L**, denoted $\lim_{n\to\infty} a_n = L$ or $a_n \to L$ as $n \to \infty$ if for every $\varepsilon > 0$ there is a positive number N such that $|a_n - L| < \varepsilon$ *whenever* $n > N$. If no such number L exists, we say that the sequence $\{a_n\}$ **has no limit**, or **diverges**

Definition: We write $\lim_{n\to\infty} a_n = \infty$ if for every positive real number $M > 0$, there is an N such that $a_n > M$ *whenever* $n > N$

Claim: $\lim_{n\to\infty} 1 + \dfrac{1}{n} = 1$. How can we prove this? One method is to do a direct proof of the ε condition. Alternatively, we can use the following Theorem:

Theorem: Let $\{a_n\}$ be a sequence given explicitly by a function $f(n)$ and suppose that $f(x)$ exists for every $x \geq 1$. Then:
(I) If $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} a_n = L$
(II) If $\lim_{x\to\infty} f(x) = \infty$, then $\lim_{n\to\infty} a_n = \infty$
(III) If $\lim_{x\to\infty} f(x) = -\infty$, then $\lim_{n\to\infty} a_n = -\infty$
Examples: Compute the limit of each of the following sequences:

1. $\left\{1 + \frac{1}{n}\right\}$

2. $\left\{n^2\right\}$

3. $\{\sin(n\pi)\}$

4. $\{\cos(n\pi)\}$

5. $\left\{\frac{n}{e^{2n}}\right\}$

Theorem:
(I) $\lim_{n\to\infty} r^n = 0$ if $|r| < 1$
(II) $\lim_{n\to\infty} r^n = \infty$ if $|r| > 1$

Note: All of our previous theorems about the algebra of limits apply to limits of sequences as well.

The Sandwich Theorem for Sequences: Given sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ with $a_n \leq b_n \leq c_n$ for every $n > N$ for some fixed N, if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Example:

Theorem: Let $\{a_n\}$ be a sequence. If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$

Definitions:
• A sequence is **monotone increasing** if for every $n \geq 1$, $a_n \leq a_{n+1}$.
• A sequence is **monotone decreasing** if for every $n \geq 1$, $a_n \geq a_{n+1}$.
(If either of these hold, a sequence is said to be **Monotonic**
• A sequence is **bounded** if there is a positive real number M such that for every $n \geq 1$, $|a_n| \leq M$.

Theorem: Every bounded monotonic sequence has a limit.