

Series Intro Part 2

Recall:

- The **harmonic series**, defined as $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.
- According to a previous Theorem, If a series $\sum a_n$ is convergent, then $\lim_{n \rightarrow \infty} a_n = 0$

Combining these two pieces of information, we have the following result:

The n th Term Test Given the series $\sum a_n$:

- (I) If $\lim_{n \rightarrow \infty} a_n \neq 0$, the the series $\sum a_n$ is divergent.
- (II) If $\lim_{n \rightarrow \infty} a_n = 0$, further investigation is needed, as the series might converge, and also might diverge.

Theorem: If $\sum a_n$ and $\sum b_n$ are series such that $a_j = b_j$ for every $j > k$, where k is a positive integer, then both series converge or both series diverge.

Proof: Since $a_n = b_n$ for $j > k$, then $\sum_{n=k+1}^{\infty} a_n = \sum_{n=k+1}^{\infty} b_n$.

Let $\sum_{n=1}^k a_n = N$ and $\sum_{n=1}^k b_n = M$.

Then, $\sum_{n=1}^{\infty} a_n = N + \sum_{n=k+1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n = M + \sum_{n=k+1}^{\infty} b_n$.

Hence either both of these converge, or both diverge.

Theorem: For any positive integer k , the series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \dots$ and $\sum_{n=k+1}^{\infty} a_n = a_{k+1} + a_{k+2} + \dots$ either both converge or both diverge.

Proof: Use the same basic argument as above (think of $\sum b_n$ where $b_n = 0$ for $n \leq k$ and $b_n = a_n$ for $n > k$).

Theorem: If $\sum a_n$ and $\sum b_n$ are both convergent series with sums A and B respectively, then:

- (I) $\sum a_n + b_n = A + B$
- (II) $\sum ca_n$ converges and has sum cA for every real number c .
- (III) $\sum a_n - b_n = A - B$

Theorem: If $\sum a_n$ is a convergent series and $\sum b_n$ is a divergent series, then $\sum a_n + b_n$ is divergent.

Proof: Suppose that there was an example of a convergent series $\sum a_n$ and a divergent series $\sum b_n$ for which the series $\sum a_n + b_n$ was convergent.

Suppose that $\sum a_n = A$ and $\sum a_n + b_n = M$.

Then, by (III) above, $\sum [(a_n + b_n) - a_n] = \sum [a_n + b_n] - \sum a_n = M - A$, so this series converges.

But $\sum [(a_n + b_n) - a_n] = \sum b_n$, which diverges by hypothesis.

Since this is clearly preposterous, our assumption that there is an example of a convergent series $\sum a_n$ and a divergent series $\sum b_n$ for which the series $\sum a_n + b_n$ was convergent must be false.

This proves the theorem. [This method of proof is called an “indirect proof”, or a “proof by contradiction”]