Taylor's Theorem

Recall: Given a function $f(x)$ with derivatives of all orders at some point c

• The **Taylor Series** generated y $f(x)$ centered at $x = c$ is: $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$

• The **Taylor Polynomial of order n** generated y $f(x)$ centered at $x = c$ is: $P_n(x) = f(c) + f'(c)(x-c) + \dfrac{f''(c)}{2!}(x-c)^2 + \cdots + \dfrac{f^{(n)}(c)}{n!}(x-c)^n$

Taylor's Theorem: Suppose f and its first n derivatives are continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there exists a number z between a and b such that $f(b) = f(a) + f'(a)(b-a) + \dfrac{f''(a)}{2!}(b-a)^2 + \cdots + \dfrac{f^{(n)(a)}}{n!}(b-a)^n + \dfrac{f^{(n+1)}(z)}{(n+1)!}(b-a)^{n+1}$.

Taylor's Formula: If a function f has derivatives of all orders in an open interval I containing a, the fro each positive integer n and each x in I, $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$, where $R_n(x) = \frac{f^{(n)}(z)}{(n+1)!}(x-a)^{n+1}$ for some z between a and x.

Note: If $R_n(x) \to 0$ and $n \to \infty$, then we say that the Taylor Series generated by f at $x = a$ *converges* to f in I.

Theorem: (Remainder Estimation) If there is a positive constant M such that $|f^{(n+1)(t)}| \leq M$ for all t between x and a, inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality:

$$|R_n(x)| \leq M\frac{|x-a|^{n+1}}{(n+1)!}$$

If this inequality holds for every n and the other conditions of Taylor's Theorem are satified by f, the the series converges to $f(x)$.

Example: Use the Taylor Series for $f(x) = e^x$ centered at $x = 0$ to approximate e to 3 decimal places of accuracy.

Recall that $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$ is the Mclaurin series for $f(x) = e^x$. From this, using Taylor's theorem, the remainder term for this series is given by:

$R_n(x) = \dfrac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$. Setting $x = 1$ and noting that $a = 0$ and $f^{(n+1)}(x) = e^x$, we then have:

$R_n(1) = \dfrac{e^z}{(n+1)!}(1-0)^{n+1} = \dfrac{e^z}{(n+1)!}(1)^{n+1} = \dfrac{e^z}{(n+1)!}$.

Note that e^x is a strictly increasing function, so it is maximized on the interval $[0, 1]$ when $z = 1$. Hence $R_n(x) \leq \frac{e}{(n+1)!}$ (since $M = e$).

Since we want our estimate to be to 3 decimal places, we need $R_n(1) \leq 0.0005$. That is, we need $\frac{e}{(n+1)!} < 0.0005$. So we must have $\frac{e}{0.0005} < (n+1)!$.

Since we do not want to depend on an accurate value of e for this computation, let's say $e < 3$, so finding n so that $(n+1)! > \frac{3}{0.0005} = 6000$ is sufficient.

Note that $7! = 5,040$, and $8! = 40,320$, so we take $n = 7$.

Therefore, we can approximate e to 3 decimal places of acuracy by adding the first 7 terms of the series:

$e \approx 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} = \frac{685}{252} \approx 2.71825$

So our estimate of e to three decimal places of accuracy is: 2.718.