Math 310
Exam 2 - Practice Problems

1. Prove that there is an integer m such that $m^3 > 10^{10}$. Is your proof constructive or non-constructive?

2. Prove that given any two rational numbers $p < q$, there is a rational number r with $p < r < q$.

3. Prove that given a non-negative integer n, there is a unique non-negative integer m such that $m^2 \leq n < (m+1)^2$

4. Prove or disprove: Every non-negative integer can be written as the sum of at most 3 perfect squares.

5. Formulate a conjecture about the final two digits of the square of any integer. Then prove your conjecture.

6. For each of the following, determine whether the statement is True or False.

   (a) $\emptyset \subseteq \{a, b, c, d\}$
   (b) $\emptyset \in \{a, b, c, d\}$
   (c) $\emptyset \in \{a, b, \emptyset\}$
   (d) $\emptyset \subseteq \{a, b, \emptyset\}$
   (e) $\{a, b\} \subset \{a, b\}$
   (f) $0 \in \{0, \{1\}, \{0, 1\}\}$
   (g) $1 \in \{0, \{1\}, \{0, 1\}\}$
   (h) $\{0, 1\} \in \{0, \{1\}, \{0, 1\}\}$
   (i) $\{0, 1\} \subset \{0, \{1\}, \{0, 1\}\}$

7. Given the set $B = \{a, b, \{a, b\}\}$ (a) Find $|B|$. (b) Find $\mathcal{P}(B)$

8. Given that $A = \{1, 2, 3\}$ and $B = \{a, b, c, d, e, f\}$

   (a) List the elements in $A \times A$.

   (b) How many elements are in $A \times B$?

   (c) How many elements are in $A \times (B \times B)$?

9. Find the set of all elements that make the predicate $Q(x) : x^2 < x$ true (where the domain of x is all real numbers).

10. Given that $A = \{0, 2, 4, 6, 8, 10, 12\}$, $B = \{0, 2, 3, 5, 7, 11, 12\}$ and $C = \{1, 2, 3, 4, 6, 7, 8, 9\}$ are all subsets of the universal set $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, find each of the following:

    (a) $A - B$
    (b) $\overline{A}$
    (c) $A \cap B$
    (d) $A \cup (B \cap C)$
    (e) $A - (\overline{B} \oplus C)$
    (f) $(A \cap C) \cup (B - \overline{A})$

11. Draw Venn Diagrams representing each of the following sets:

    (a) $A - B$
    (b) $B - \overline{A}$
    (c) $(A \cup C) \cap B$
    (d) $\overline{A \cup B \cup C}$
    (e) $A - (B \cup C)$
    (f) $(A \cap B) - \overline{C}$

12. Use a membership table to show that $(B - A) \cup (C - A) = (B \cup C) - A$.

13. Use a 2-column proof to verify the set identity: $A \cup (A \cap B) = A$.

14. Use a paragraph (double containment) proof to show that $A - B = A \cap \overline{B}$.

15. For each of the following, either prove the statement or show that it is false using a counterexample.

    (a) $(A - B) - C = A - (B - C)$
    (b) $A \oplus (B \oplus C) = (A \oplus B) \oplus C$
    (c) $A \cap (B - C) = (A \cap B) - (A \cap C)$

16. Consider the function $f(x) = |x|$

    (a) Suppose that the domain of this function is $\mathbb{R}$ and the co-domain is $\mathbb{R}$. Find the range of f. Is f 1-1? Is f onto? Justify your answers.
    (b) Suppose that the domain of this function is $\mathbb{N}$ and the co-domain is $\mathbb{N}$. Find the range of f. Is f 1-1? Is f onto? Justify your answers.
    (c) Suppose $S = \{-2, -1, 0, 1, 2\}$. Find $f(S)$ (the image of the set S under f). Find $f^{-1}(S)$ (the preimage of the set S under f).

17. For each of the following functions, determine whether f is a one-to-one. Also determine whether f is onto. Justify your answers.

    (a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - x$
    (b) $f : \mathbb{R}^+ \to \mathbb{R}^+$ $f(x) = x^2$
    (c) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ $f(m, n) = m^2 - n$
    (d) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ $f(m, n) = m^2 - n^2$

18. Prove or Disprove: Suppose $f : B \to C$ and $g : A \to B$. If f is one-to-one and g is onto, then $f \circ g$ is one to one.

19. Prove or Disprove: Suppose $f : B \to C$ and $g : A \to B$. If f is one-to-one and g is onto, then $f \circ g$ is onto.

20. Prove that $n^5 - n$ is divisible by 5 for any non-negative integer n.

21. Prove that for $r \in \mathbb{R}$, $r \neq 1$ and for all integers n, $\sum_{j=0}^{n} r^j = \dfrac{r^{n+1} - 1}{r - 1}$

22. Prove that for all $n \geq 2$, $\sum_{k=1}^{n} \dfrac{1}{k^2} < 2 - \dfrac{1}{n}$

23. Prove that $n! < n^n$ whenever $n > 1$.

24. Prove that for all n, $\sum_{k=1}^{n} \dfrac{1}{(2k-1)(2k+1)} = \dfrac{n}{2n+1}$

25. Suppose that $f(x) = e^x$ and $g(x) = xe^x$. Use induction and the product rule to show that $g^{(n)}(x) = (x + n)e^x$ for all $n \geq 1$.

26. Given the relation $R = \{(1, 1), (1, 3), (1, 4), (2, 2), (3, 1), (3, 4), (4, 1), (4, 3)\}$ on the set $A = \{1, 2, 3, 4\}$:

    (a) Determine whether or not R is reflexive.
    (b) Determine whether or not R is irreflexive.
    (c) Determine whether or not R is symmetric.
    (d) Determine whether or not R is antisymmetric.
    (e) Determine whether or not R is transitive.

27. Given the relation $S = \{(1, 1), (1, 3), (2, 1), (2, 2), (2, 3), (3, 3), (4, 4)\}$ on the set $A = \{1, 2, 3, 4\}$:

    (a) Determine whether or not S is reflexive.
    (b) Determine whether or not R is irreflexive.
    (c) Determine whether or not R is symmetric.
    (d) Determine whether or not R is antisymmetric.
    (e) Determine whether or not R is transitive.

28. Suppose that R and S are symmetric relations on a non-empty set A. Prove or disprove each of these statements:

    (a) $R \cup S$ is symmetric.
    (b) $R \cap S$ is symmetric.
    (c) $R - S$ is symmetric.
    (d) $R \oplus S$ is symmetric.
    (e) $S \circ R$ is symmetric.

29. Given the relation $R = \{(1, 1), (1, 3), (1, 4), (2, 2), (3, 1), (3, 4), (4, 1), (4, 3)\}$ on the set $A = \{1, 2, 3, 4\}$:

    (a) Find the matrix representation M_R for this relation. (b) Draw the graph representation of this relation Γ_R.

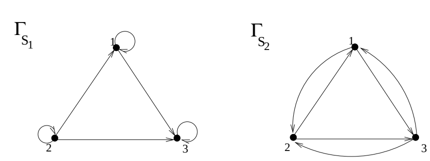
30. Given the relation $S = \{(1, 1), (1, 3), (2, 1), (2, 2), (2, 3), (3, 3), (4, 4)\}$ on the set $A = \{1, 2, 3, 4\}$:

    (a) Find the matrix representation M_S for this relation. (b) Draw the graph representation of this relation Γ_S.

31. Let R_1 and R_2 be given by the matrices $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

    (a) Determine whether or not R_1 is reflexive, irreflexive, symmetric, antisymmetric, transitive.
    (b) Determine whether or not R_2 is reflexive, irreflexive, symmetric, antisymmetric, transitive.
    (c) Find the matrices representing $\overline{R_1}$, $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 \oplus R_2$, and $R_1 \circ R_2$
    (d) Draw the graphs representing R_1 and R_2.

32. Given the graphs representing the relations S_1 and S_2:



    (a) Determine whether or not R_1 is reflexive, irreflexive, symmetric, antisymmetric, transitive.
    (b) Determine whether or not R_2 is reflexive, irreflexive, symmetric, antisymmetric, transitive.
    (c) Draw the graph representing $\overline{S_2}$, $S_1 \cup S_2$, $S_1 \cap S_2$, $S_2 - S_1$, and $S_2 \circ S_1$
    (d) Find the matrix representing S_1.
    (e) List the ordered pairs in S_2.