Math 310
Paths and Connectivity

Definitions: Let $\Gamma = (V, E)$ be an undirected graph and let u and v be (not necessarily distinct) vertices in Γ.

• A **path of length** n from u to v in Γ is a sequence $\langle v_0, e_1, v_1, e_2, ..., v_{n-2}, e_{n-1}, v_{n-1}, e_n, v_n\rangle$ with $v_0 = u$, $v_n = v$, and each edge e_i for $1 \leq i \leq n$ connects v_{i-1} and v_i.
• Given such a path, the sequence $\langle v_0, v_1, ..., v_{n-1}, v_n\rangle$ is the **vertex sequence** of the path, and the sequence $\langle e_1, e_2, ..., e_{n-1}, e_n\rangle$ is the **edge sequence** of the path.
• A path is said to **pass through** the vertices $v_1, v_2, ..., v_{n-1}$. v_0 and v_n are called the **endpoints** of the path. A path is said to **traverse** its edges $e_1, e_2, ..., e_n$.
• In a simple graph, we may use the vertex sequence for a path to identify the path, but in a multigraph, the vertex sequence may not give enough information to completely specify the path.
• A path is a **circuit** if $v_0 = v_n$ (that is, if it begins and ends at the same vertex) and the length of the path is greater than zero.
• A path (or circuit) is **simple** if it does not contain the same edge more than once.
• We say that Γ is **connected** if there is at least one path between every (distinct) pair of vertices in V.
• A **connected component** of a graph Γ is a proper subgraph of Γ that is connected and which is not properly contained in any other connected subgraph of Γ. That is, it is a maximal connected proper subgraph of Γ.

Definitions: Let $\Gamma = (V, E)$ be a directed graph and let u and v be (not necessarily distinct) vertices in Γ.

• A **path of length** n from u to v in Γ is a sequence $\langle v_0, e_1, v_1, e_2, ..., v_{n-2}, e_{n-1}, v_{n-1}, e_n, v_n\rangle$ with $v_0 = u$, $v_n = v$, and each edge e_i for $1 \leq i \leq n$ connects v_{i-1} to v_i (as a directed edge).
• A (directed) path is a **circuit** if $v_0 = v_n$ (that is, if it begins and ends at the same vertex) and the length of the path is greater than zero.
• A path (or circuit) is **simple** if it does not contain the same edge more than once.
• A directed graph Γ is **strongly connected** if for every pair of vertices u, v in Γ, there is a path from u to v and there is also a path from v to u.
• A directed graph Γ is **weakly connected** if there its underlying undirected graph is connected.

Theorem: There is a simple path between every pair of distinct vertices in a connected undirected graph.

Proof: Let u and v be distinct vertices in a connected undirected graph Γ. By the definition of connected, there is at least on path from u to v. Let $\langle v_0, v_1, ..., v_{n-1}, v_n\rangle$ be the vertex sequence of a path from u to v of shortest possible length.
Claim: this shortest path is simple.
To see this, suppose that it is not simple. Then $v_i = v_j$ for some pair of vertices in the vertex sequence with $0 \leq i < j$ (this is because whenever there is a repeated edge, at least one of the vertices along that edge must be visited twice). But then we may remove a segment of the path to get a path of shorter length with vertex sequence $v_0, v_1, ...v_{i-1}, v_j, v_{j+1}, ..., v_n$. If this were the case, we would have found a path from u to v that is shorter than our original path, contradicting our original assumption. □

Definitions:
• An **Euler path** in a graph Γ is a simple path containing every edge of Γ. An *Euler circuit* is an Euler path that is a circuit (i.e that begins and ends at the same vertex).
• A **Hamiltonian path** in a graph Γ is a simple path that passes through every vertex of Γ exactly once. A **Hamiltonian circuit** is a Hamiltonian path that is a circuit (i.e that begins and ends at the same vertex).