Math 310
Partial Orderings Handout

**Definitions:**

- A relation $R$ on a set $S$ is a **partial ordering** or **partial order** if it is reflexive, antisymmetric, and transitive.
- A set $S$ together with a partial ordering $R$ is called a **partial ordered set** or **poset**, denoted by $(S, R)$ or $(S, \preceq)$.
- Members of the set $S$ are called **elements** of the poset.
- The elements $a$ and $b$ of a poset $(S, \preceq)$ are called **comparable** if either $a \preceq b$ or $b \preceq a$. If neither of these occur for a pair $a$ and $b$, then we say that these elements are **incomparable**.
- If $(S, \preceq)$ is a poset and every pair of elements in $S$ are comparable, then we say that $S$ is a **totally ordered set** and we say that $\preceq$ is a **total order**.
- If $S$ is a totally ordered set with total ordering $\preceq$ in which every non-empty set has a least element, then we say that $(S, \preceq)$ is a **well-ordered set**.

**Examples:**

1. $(\mathbb{R}, \geq)$ is a poset.
2. $(\mathbb{Z}, \leq)$ is a poset.
3. $(\mathbb{Z}^+, |)$ is a poset.
4. Given a set $A$, $(\mathcal{P}(A), \subseteq)$ is a poset.
5. Claim: $(\mathbb{Z}^+, \leq)$ is a well-ordered set.
6. Claim: $(\mathbb{R}, \leq)$ is not a well-ordered set.
7. Extra Credit: Find an order $\preceq$ on $\mathbb{R}$ that makes $(\mathbb{R}, \preceq)$ a well ordered set.

**Lexicographic Ordering:** Given two posets $(A_1, \preceq_1)$ and $(A_2, \preceq_2)$, the **lexicographic ordering** $\preceq$ on $A_1 \times A_2$ is defined as follows:

Given two elements $(a_1, a_2)$ and $(b_1, b_2)$, the pair $(a_1, a_2) \preceq (b_1, b_2)$ if either $a_1 \preceq_1 b_1$ or $a_1 = b_1$ and $a_2 \preceq_2 b_2$. Note that $(a_1, a_2) = (b_1, b_2)$ if $a_1 = b_1$ and $a_2 = b_2$.

**Example:** Consider $(\mathbb{Z}^+ \times \mathbb{Z}^+, \preceq)$ where we use the standard $\leq$ ordering on each component.

(a) Notice that $(3, 17) \preceq (4, 11)$ since $3 \leq 4$.

(b) Similarly, $(5, 2) \preceq (5, 7)$ since $5 = 5$ and $2 \leq 7$.

**Note:** We can expand this definition to $n$-tuples of elements by applying the same process of comparing the elements in coordinates working from left to right. For example, $(2, 12, 35) \preceq (3, 1, 4)$, $(3, 2, 17, 11) \preceq (3, 5, 8, 1)$, and $(2, 5, 8, 11, 17) \preceq (2, 5, 8, 11, 22)$ in the lexicographic orderings induced by using $\leq$ in each component. We can get more exotic lexicographic orderings by applying different orderings in one or more of the components.

**More Definitions:**

- An element $a$ in a poset $(S, \preceq)$ is **maximal** if these is no element $b$ such that $a \prec b$. Similarly, an element $a$ in a poset $(S, \preceq)$ is **minimal** if these is no element $b$ such that $b \prec a$.
- An element $a$ in a poset $(S, \preceq)$ is the **greatest element** if $b \preceq a$ for all $b \in S$. Similarly, an element $a$ is the **least element** if $a \preceq b$ for all $b \in S$
- Given a poset $(S, \preceq)$ and a subset $A \subseteq S$, an element $u$ is an **upper bound** for $A$ if $a \preceq u$ for all $a \in A$. An element $x$ is the **least upper bound** of $A$ if $x$ is an upper bound of $A$, and given any upper bound $u$ of $A$, $x \preceq u$.
- Given a poset $(S, \preceq)$ and a subset $A \subseteq S$, an element $l$ is a **lower bound** for $A$ if $l \preceq a$ for all $a \in A$. An element $y$ is the **greatest lower bound** of $A$ if $y$ is a lower bound of $A$, and given any lower bound $l$ of $A$, $l \preceq y$.
- A poset in which every pair of elements has both a least upper bound and a greatest lower bound is called a **lattice**.

**Example:**