Math 310
Introduction to Proofs

Terminology

- A **theorem** is a statement that can be shown to be true (we usually reserve this term for "important" results)
- Less important statements that can be shown to be true are often called **propositions**.
- A **proof** is a valid argument that establishes the truth of a statement.
- An **axiom** (or **postulate**) is a statement that we assume to be true.
- The statements used in a proof can include axioms and previously proved theorems or propositions.
- A **lemma** is a statement that, although it may not be important on its own, is helpful in proving other results.
- A **corollary** is a theorem that can be established directly from a previous theorem.
- A **conjecture** is a statement that is being proposed as possibly true based upon partial evidence.

Proof Methods

1. Direct Proofs

   A direct proof is used when we want to prove a theorem of the form: $\forall x\,(P(x) \rightarrow Q(x))$ and we are able to show that the conclusion holds whenever the hypothesis holds. The method proceeds as follows:

   • Begin by assuming that the hypothesis $P(a)$ holds for an arbitrary element a in the domain of the variable x.

   • Use axioms, definitions, and previously proved theorems together with the rules of inference to show that the conclusion $Q(a)$ is also true.

   • Since we have now shown that $P(a) \rightarrow Q(a)$ for arbitrary a, we conclude that $\forall x\,(P(x) \rightarrow Q(x))$ is a true statement using universal generalization.

2. Proof by Contraposition

   Proof by Contraposition is used when we want to prove a theorem of the form: $\forall x\,(P(x) \rightarrow Q(x))$ and it seems easier to show that the hypothesis fails to hold whenever the conclusion fails to hold. The method proceeds as follows:

   • Begin by assuming $\neg Q(a)$ (that is, that the conclusion $Q(a)$ does not hold) for an arbitrary element a in the domain of the variable x.

   • Use axioms, definitions, and previously proved theorems together with the rules of inference to show that $\neg P(a)$ is also true.

   • Since we have now shown that $\neg Q(a) \rightarrow \neg P(a)$ for arbitrary a, we conclude that $\forall x\,(P(x) \rightarrow Q(x))$ is a true statement using contraposition and universal generalization.

3. Proof by Contradiction

   In Proof by Contradiction, we show that a statement p must be true by showing that its negation *cannot* be true. That is, we show that assuming that $\neg p$ is true leads to a contradiction ($\neg p \rightarrow (r \wedge \neg r)$ for some statement r). We tend to use this method only when more direct methods do not work easily. The method proceeds as follows:

   • Begin by assuming that the statement $\neg p$ holds.

   • Use axioms, definitions, and previously proved theorems together with the rules of inference to show that whenever $\neg p$ holds, then r and its negation $\neg r$ both hold.

   • Since it is impossible for r and $\neg r$ to be true at the same time, we conclude that $\neg p$ cannot be true, and thus that p must be true (note that p must be a statement for this to work).

4. If and Only If Proofs

   To prove a theorem of the form $\forall x\,(P(x) \leftrightarrow Q(x))$, we usually proceed by proving the two related conditional statements separately. That is we proceed as follows:

   • Use some valid proof method to prove the theorem $\forall x\,(P(x) \rightarrow Q(x))$

   • Use some valid proof method to prove the theorem $\forall x\,(Q(x) \rightarrow P(x))$

   • Conclude that the original theorem $\forall x\,(P(x) \leftrightarrow Q(x))$ is true.

   Note: In this case, we say that we have shown the the two statements $P(x)$ and $Q(x)$ are *equivalent*.

5. Vacuous Proofs

   A theorem of the form $\forall x\,(P(x) \rightarrow Q(x))$ is shown to be **vacuously true** by proving that $P(x)$ is false for all x in the domain of the variable. The method proceeds as follows:

   • Use axioms, definitions, and previously proved theorems together with the rules of inference to show that the hypothesis $P(a)$ is false for an arbitrary element a.

   • Conclude that $\forall x\,(P(x) \rightarrow Q(x))$ is a true statement using universal generalization and the fact that a conditional statement is false only when the hypothesis is true and the conclusion is false.

   The idea here is that the Theorem is true, but only for a very silly reason – the fact that the hypothesis of the theorem can never be met.

6. Trivial Proofs

   A theorem of the form $\forall x\,(P(x) \rightarrow Q(x))$ is shown to be **trivially true** by proving that $Q(x)$ is true for all x in the domain of the variable. The method proceeds as follows:

   • Use axioms, definitions, and previously proved theorems together with the rules of inference to show that the conclusion $Q(a)$ is true for an arbitrary element a.

   • Conclude that $\forall x\,(P(x) \rightarrow Q(x))$ is a true statement using universal generalization and the fact that a conditional statement is false only when the hypothesis is true and the conclusion is false.

   The idea here is that the Theorem is true, but only for a very silly reason – the fact that the conclusion always holds, regardless of whether the hypothesis is satisfied, so the conclusion does not depend of the hypothesis being met.

Common Errors in Proofs

1. Invalid Steps

   A proof fails to be valid if even one of its steps is untrue. Each step in the proof must be able to be verified using axioms, previously proved theorems, definitions, and rules of inference. Some false steps can be difficult to see because they are false of subtle reasons.

2. Begging the Question (or Circular Reasoning)

   A proof fails to be valid if one of its steps unwittingly assumes that the result being proved is true.