Exercise 3.83. Find a matrix of the reflection that maps the line *l*[2, 3, –1] to *m*[2, 3, 5].

Method 1.

First, note that the axis of reflection must be *p*[2, 3, 2].

We find the point of intersection of line *h* and *p*, by applying Proposition 3.2.

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ 0 & 1 & 0 \\ 2 & 3 & 2 \end{vmatrix} = 0$$

Hence, $-u_1 + u_3 = 0.$ Thus, the point of intersection *C* is (–1, 0, 1).

The measure of the angle between lines *h* and *p* is:

$$m\angle(h,p) = \tan^{-1}\left(\frac{0(3)-1(2)}{0(2)+1(3)}\right) = \tan^{-1}\left(-\frac{2}{3}\right).$$

Notice that

$$\cos\left(\tan^{-1}\left(-\frac{2}{3}\right)\right) = \cos\left(\tan^{-1}\left(\frac{2}{3}\right)\right) = \frac{1}{\sqrt{1+\tan^2\left(\tan^{-1}\left(\frac{2}{3}\right)\right)}} = \frac{3}{\sqrt{13}}$$

and

$$\sin\left(\tan^{-1}\left(-\frac{2}{3}\right)\right) = -\sin\left(\tan^{-1}\left(-\frac{2}{3}\right)\right) = -\sqrt{1-\cos^2\left(\tan^{-1}\left(-\frac{2}{3}\right)\right)} = -\frac{2}{\sqrt{13}}.$$

Hence, by Proposition 3.15,

$$R_p = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} & 0 \\ -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{13}} & 0 \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{13} & -\frac{12}{13} & -\frac{8}{13} \\ -\frac{12}{13} & -\frac{5}{13} & -\frac{12}{13} \\ 0 & 0 & 1 \end{bmatrix}.$$

We check the solution by applying Proposition 3.6.

$$kmR_p = k\begin{bmatrix} 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} \frac{5}{13} & -\frac{12}{13} & -\frac{8}{13} \\ -\frac{12}{13} & -\frac{5}{13} & -\frac{12}{13} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -3 & 1 \end{bmatrix}.$$

The result checks when *k* = 1.

Method 2.

First, note that the axis of reflection must be *p*[2, 3, 2]. We apply Proposition 3.6 to find the matrix of a direct isometry *T* that maps *h*[0, 1, 0] to *p*[2, 3, 2].

There is a nonzero real number *k* such that *kpT* = *h*.

$$kpT = k\begin{bmatrix}2 & 3 & 2\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\end{bmatrix} = h$$

Which implies $k\begin{bmatrix}2\cos\theta+3\sin\theta & -2\sin\theta+3\cos\theta & 2a+3b+2\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\end{bmatrix}$.

Solving this system, using the fact that det(*T*) = 1, we obtain: $k = \frac{1}{\sqrt{13}}$, $\cos\theta = \frac{3}{\sqrt{13}}$, $\sin\theta = -\frac{2}{\sqrt{13}}$, and $b = -\frac{1}{3}(2a+2)$.

For a matrix of a direct isometry that maps line *h* to line *l*, we let *a* = 2 and *b* = –2,

$$T = \begin{bmatrix}\frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} & 2 \\ -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} & -2 \\ 0 & 0 & 1\end{bmatrix}.$$

Hence, R_p is defined by

$$R_p = \begin{bmatrix}\frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} & 2 \\ -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} & -2 \\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{13}} & -\frac{10}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}\frac{5}{13} & -\frac{12}{13} & -\frac{8}{13} \\ -\frac{12}{13} & -\frac{5}{13} & -\frac{12}{13} \\ 0 & 0 & 1\end{bmatrix}.$$

We check the solution by applying Proposition 3.6.

$$kmR_p = k\begin{bmatrix}2 & 3 & 5\end{bmatrix}\begin{bmatrix}\frac{5}{13} & -\frac{12}{13} & -\frac{8}{13} \\ -\frac{12}{13} & -\frac{5}{13} & -\frac{12}{13} \\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}-2 & -3 & 1\end{bmatrix}.$$

The result checks when *k* = 1.