Math 127
Exponential Functions

A. Definition: An **exponential function** is a function of the form $f(x) = a^x$ for $0 < a < 1$ or $a > 1$.
Note: a is called the **base** of the exponential function. As we will see, exponential functions have domain $(-\infty, \infty)$ and range $(0, \infty)$.

Note: The reason we exclude 0 and 1 as bases for exponential function is because $0^x = 0$ for and x, and $1^x = 1$ for any x, so these are just constant functions.

Notice that these functions are quite different from other functions we have looked at so far. Here, the *exponent* part of the expression defining the function is a *variable*.

Example 1: Let $f(x) = 3^x$. Then:
(a) $f(0) = 3^0 = 1$

(b) $f(2) = 3^2 = 9$

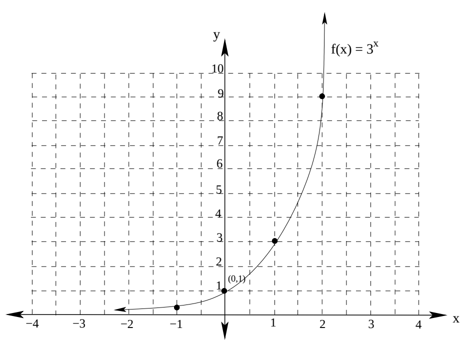
(c) $f(-3) = 3^{-3} = \frac{1}{27}$

(d) $f(\frac{2}{3}) = 3^{\frac{2}{3}} = \sqrt[3]{3^2} = \sqrt[3]{9} \approx 2.080084$

Graphs of exponential functions:

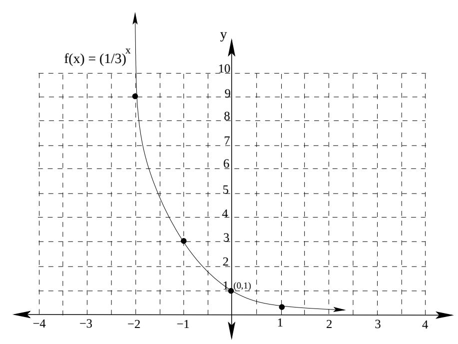
$f(x) = 3^x$

x	$f(x)$
-3	$\frac{1}{27}$
-2	$\frac{1}{9}$
-1	$\frac{1}{3}$
0	1
1	3
2	9
3	27



$f(x) = \left(\frac{1}{3}\right)^x$

x	$f(x)$
3	$\frac{1}{27}$
2	$\frac{1}{9}$
1	$\frac{1}{3}$
0	1
-1	3
-2	9
-3	27



Facts:

(1) If $f(x) = a^x$ with $a > 1$, then f is an increasing function, and hence is a one-to-one function.

(2) If $f(x) = a^x$ with $0 < a < 1$, then f is a decreasing function, and hence is a one-to-one function.

Solving Basic Exponential Equations:

We can use the fact that exponential functions are one-to-one to solve various equations involving exponentials. This is because we can make use of the fact that if $a^{x_1} = a^{x_2}$, then $x_1 = x_2$.

Examples:

1. $4^{2x-3} = 4^{5-x}$

Since $f(x) = 4^x$ is a one-to-one function, we can conclude that:

$2x - 3 = 5 - x$, or $3x = 8$.

Hence $x = \frac{8}{3}$.

2. $2^{4x-7} = 8^{2x-5}$

Since $8 = 2^3$, we can rewrite 8^{2x-5} as $\left(2^3\right)^{2x-5} = 2^{3(2x-5)} = 2^{6x-15}$.

Then, as above, we know that $4x - 7 = 6x - 15$, or $8 = 2x$.

Hence $4 = x$.

Compound Interest

There are many practical applications for exponential functions. One of the most common is computing compound interest.

The Compound Interest Formula: When a principal amount P in invested at interest rate r which is compounded n times per year and remains invested for t year, the amount A that results is given by the formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$

Examples:

1. Suppose you put \$1000 in an account that pays 6% interest compounded monthly. How much money will be in the account 3 years later?

   $P = 1000$, $r = 0.06$, $n = 12$, and $t = 3$, so $A = 1000\left(1 + \frac{.06}{12}\right)^{(12)(3)} = 1000\,(1.005)^{36} \approx \$1,196.68$

2. Now Suppose you put \$2000 in an account that pays 7% interest compounded daily. How much money will be in the account 5 years later?

   $P = 2000$, $r = 0.07$, $n = 365$, and $t = 5$, so $A = 2000\left(1 + \frac{.07}{365}\right)^{(365)(5)} \approx 2000\,(1.000191781)^{1825} \approx \$2838.04$

The Natural Exponential Function:

Definition: If we consider what happens to the base of our compound interest exponential term: $\left(1 + \frac{1}{n}\right)$ as we compound more and more frequently.

n	$\left(1 + \frac{1}{n}\right)^n$
1	2.0
10	2.59374246
100	2.704813829
1,000	2.71692393
10,000	2.71814593
100,000	2.71826824
1,000,000	2.71828047

As n gets bigger and bigger, $\left(1 + \frac{1}{n}\right)$ approaches an irrational number we call e, **the base of the natural exponential function**, $f(x) = e^x$.

Since $e \approx 2.7182818$, $2^x < e^x < 3^x$.

Continuously Compounded Interest Using this new base, we can measure the accumulation of interest that is compounded "instantaneously" rather than only n times a year. We do so using the formula: $A = Pe^{rt}$, where P, A, r, and t are exactly as above.

Example: Suppose you invest \$1000 at 6% interest compounded continuously for 3 years. Then at the end of the 3 years, you will have: $1000e^{0.06(3)} \approx \$1,197.22$

Notice that this is about 54 cents more that we had investing the same amount at the same interest rate but only compounded monthly.

Example: Suppose the population of a bacterial colony if given by the function $f(t) = 500e^{-.87t}$ where t is in hours and $f(t)$ is in thousands of cells.

Then $f(0) = 500e^{-.087(0)} = 500e^0 = 500$, so there are initially 500,000 cells in the colony.

Similarly, $f(5) = 500e^{-.087(5)} = 500e^{-0.435} \approx 323.632$, so after 5 hours, the population of the colony has been reduced to 323,632 cells.