Math 127
Properties of Logarithms

**Recall:** The **Logarithm of $x$ to the base $b$** is defined as follows: $y = \log_b x$ if and only if $x = b^y$. for $x > 0$ and $b > 0, b \neq 1$. A logarithm basically asks: "what power would I need to raise the base $b$ to in order to get $x$ as the result?"

**Properties of logarithms:** Let $m$ and $n$ be positive real numbers.

1. $\log_b mn = \log_b m + \log_b n$
2. $\log_b \frac{m}{n} = \log_b m - \log_b n$
3. $\log_b m^n = n \cdot \log_b m$
4. $\log_b 1 = 0$
5. $\log_b b = 1$
6. $\log_b b^x = x$
7. $b^{\log_b x} = x$

**Examples:** Use the Properties of Logarithms to expand the following:

1. $\log_b 16 = \log_b 2^4 = 4 \log_b 2$
2. $\log_b \frac{7}{16} = \log_b 7 - \log_b 16 = \log_b 7 - \log_b 2^4 = \log_b 7 - 4 \log_b 2$
3. $\log_b \left( \dfrac{(x+4)^3(x-1)^2}{\sqrt{x+1}} \right)$
   $= \log_b \left((x+4)^3(x-1)^2\right) - \log_b \left(\sqrt{x+1}\right)$
   $= \log_b (x+4)^3 + \log_b (x-1)^2 - \log_b (x+1)^{\frac{1}{2}}$
   $= 3 \log_b (x+4) + 2 \log_b (x-1) - \dfrac{1}{2} \log_b (x+1)$

**Example:** Use the Properties of Logarithms to combine the following into a single logarithm:

$= \dfrac{5}{2} \log_b (2x-7) - \log_b (3x+1) - \dfrac{3}{2} \log_b (x+1)$

$= \dfrac{5}{2} \log_b (2x-7) - [\log_b (3x+1) + \dfrac{3}{2} \log_b (x+1)]$

$= \log_b (2x-7)^{\frac{5}{2}} - [\log_b (3x+1) + \log_b (x+1)^{\frac{3}{2}}]$

$= \log_b (2x-7)^{\frac{5}{2}} - [\log_b (3x+1)(x+1)^{\frac{3}{2}}]$

$= \log_b \left( \dfrac{(2x-7)^{\frac{5}{2}}}{\log_b (3x+1) + \log_b (x^1+1)^{\frac{3}{2}}} \right)$

**Examples:** Solving Logarithmic Equations:

1. $\log_3 (x+6) - \log_3 (x-2) = 2$

Then $\log_3 \left( \dfrac{x+6}{x-2} \right) = 2$, so $3^2 = \dfrac{x+6}{x-2}$

Therefore, $9(x-2) = x+6$, or $9x - 18 = x + 6$. Hence $8x = 24$, or $x = 3$

Check: $\log_3 (3+6) - \log_3 (3-2) = \log_3 (9) - \log_3 (1) = 2 - 0 = 2$

2. $\ln x = 1 - \ln(3x-2) - \ln e$

Then $\ln x + \ln(3x-2) = 1 - 1$, or $\ln(x(3x-2)) = 0$

But then, exponentiating both sides: $e^{\ln(x(3x-2))} = e^0$, or $x(3x-2) = 1$

Thus $3x^2 - 2x - 1 = 0$, or $(3x+1)(x-1) = 0$.

Hence $3x = -1$, or $x = -\frac{1}{3}$ and $x = 1$

Notice that $x = -\frac{1}{3}$ does not check while $x = 1$ does check.