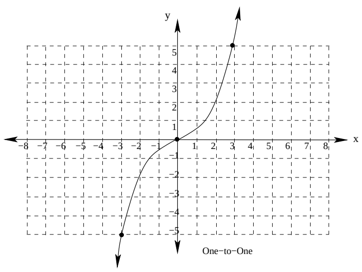
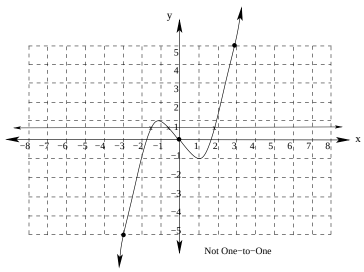
Math 127
One-to-One Functions and Inverse Functions

A. Definition: A function f is a **one-to-one function** if either of the following equivalent conditions is satisfied:
(1) Whenever $a \neq b$ in D, then $f(a) \neq f(b)$ in R.
(2) Whenever $f(a) = f(b)$ in R, then $a = b$ in D.

The first version of the definition emphasizes the fact that in a one-to-one function, different x values are mapped to different y values.
The second version of the definition emphasizes the fact that if the function ever outputs the same y value, then the inputs used must have the same value.
With this in mind, if we consider graphs of functions, we have the following result:

The Horizontal Line Test: A function f is one-to-one function if and only if every horizontal line intersects the graph of f in at most one point.





B. Determining Whether or Not a Given Function is One-to-One.

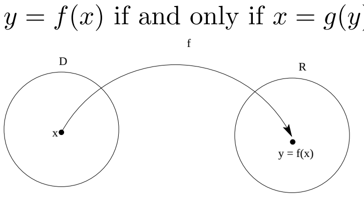
Example 1:
$f(x) = 4x - 3$ is one-to-one since if $f(x_1) = f(x_2)$, then $4x_1 - 3 = 4x_2 - 3$
But then $4x_1 = 4x_2$, so $x_1 = x_2$.

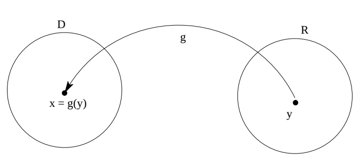
Example 2:
$g(x) = x^2$ is not one-to-one since if $x_1 = -2$ and $x_2 = 2$, then $g(-2) = g(2) = 4$.

C. Increasing and Decreasing Functions.
Theorem:
(1) A function that is increasing throughout its domain is one-to-one.
(2) A function that is decreasing throughout its domain is one-to-one.

Proof Sketch:
Notice that functions that are always increasing can never output the same value twice since to do so would require them to either decrease or remain constant.
Similarly, functions that are always decreasing can never output the same value twice since to do so would require them to either increase or remain constant.

D. Inverse Functions.
Definition: Let f be a one-to-one function with domain D and range R. A function g with domain R and range D is the **inverse function** of f, provided the following conditions are true for every x in D and y in R:
$y = f(x)$ if and only if $x = g(y)$.





Note: If a function f is not one-to-one, it cannot have a well defined inverse function, since if there is are distinct x values x_1 and x_2 with $f(x_1) = f(x_2)$, then when we attempt to define $g(y)$, we must have both $g(y) = x_1$ and $g(y) = x_2$, which violates the definition of a function!

A consequence of this definition is the following result:

Theorem on Inverse Functions: Let f be a one-to-one function with domain D and range R. If g is a function with domain R and range D, then g is the inverse function of f if and only if both of the following are true:
(1) $g(f(x)) = x$ for every x in D
(2) $f(g(y)) = y$ for every y in R

Notation: Often, when a function f has an inverse function, we use the symbol f^{-1} to denote the inverse function of f.

Warning!!! $f^{-1} \neq \frac{1}{f(x)}$. Because of this, using f^{-1} is potentially confusing, but it's the best we've got.

E. Finding the Inverse of a Function.

Example 1: $f(x) = 4x - 3$
(1) Set $y = f(x)$ and use algebra to solve for x in terms of y:
$y = 4x - 3$, so $y + 3 = 4x$.
Then $\frac{y+3}{4} = x$, or $x = \frac{1}{4}y + \frac{3}{4}$

(2) Switch variables and rewrite as an inverse function:
$f^{-1}(x) = \frac{1}{4}x + \frac{3}{4}$

(3) Check to see if the composition property of inverse functions is satisfied:
$f(f^{-1}(x)) = 4(\frac{1}{4}x + \frac{3}{4}) - 3 = x + 3 - 3 = x$
$f^{-1}(f(x)) = \frac{1}{4}(4x - 3) + \frac{3}{4} = x - \frac{3}{4} + \frac{3}{4} = x$

Example 2: $f(x) = \frac{2x}{x+3}$
(1) Set $y = f(x)$ and use algebra to solve for x in terms of y:
$y = \frac{2x}{x+3}$, so $y(x+3) = 2x$, or $xy + 3y = 2x$
$xy - 2x = -3y$, so $x(y-2) = -3y$
Hence $x = -\frac{3y}{y-2}$

(2) Switch Variables and Rewrite as an Inverse Function:
$f^{-1}(x) = -\frac{3x}{x-2}$

(3) Check to see if the composition property of inverse functions is satisfied:

F. Graphs of Inverse Functions.
A consequence of the fact that if $f(x) = y$, then $f^{-1}(y) = x$ is that whenever a point (x, y) is on the graph of a function, the point (y, x) is on the graph of the unverse function f^{-1}. As a result, the graph of f^{-1} is the graph of f reflected across the line $y = x$. Moreover, the Domain of f is the Range of f^{-1}, and the Range of f is the Domain of f^{-1}.