Math 487
Quiz 1

Instructions: You will have 55 minutes to complete this exam. The credit given on each problem will be proportional to the amount of correct work shown. Answers without supporting work will receive little credit.
Work your exam on separate sheets of paper. Be sure to put your name on at least the front page.

1. (4 points each) Explain each of the following in your own words:

   (a) What is required for an axiomatic system to be consistent?

   An axiomatic system is *consistent* if there is no statement such that both the statement and its negation are axioms or theorems of the axiomatic system.

   (b) What is required for an axiomatic system to be complete?

   An axiomatic system is *complete* if every statement about the undefined and defined terms of the system can be proved to be either valid or invalid.

2. (3 points each) Determine whether There is a projective plane with exactly the number of points and lines indicated. Justify your answers.

   (a) 100 points and 100 lines

   Recall that according to Theorem 5 for Finite Projective Geometries: In a projective plane of order n, there exist exactly n^2+n+1 points and n^2+n+1 lines.

   Notice that if $n = 9$, then $n^2+n+1 = 9^2+9+1 = 81+9+1 = 91$
   Similarly, if $n = 10$, then $n^2+n+1 = 10^2+10+1 = 100+10+1 = 111$

   Since $91 < 100 < 111$, there is no positive integer n such that $n^2+n+1 = 100$, hence there is no finite projective plane with exactly 100 points and 100 lines.

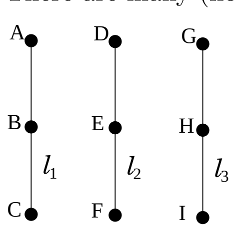
   (b) 91 points and 91 lines

   From above, if $n = 9$, then $n^2+n+1 = 9^2+9+1 = 81+9+1 = 91$, so Theorem 5 is not violated. However, this, in itself, is not enough to guarantee that there is a projective place of this "size".
   However, by Oswald Veblen and W. Bussey, projective planes exist for n such that $n = p^m$, and $9 = 3^2$, so 9 is a power of a prime, and hence a projective plane of order 9 exists, thus a projective plane with exactly 91 points and 91 lines exists.

3. Consider the following axiomatic system:
   $A1$: Each line is incident with exactly three points
   $A2$: There are exactly three lines
   $A3$: Every pair of distinct points is incident with *at most* one line
   $A4$: Every pair of distinct lines has *at most* one point in common.

   (a) (6 points) Find a model for this geometry.

   There are many (non-isomorphic) models for this geometry. Here is one possible model:

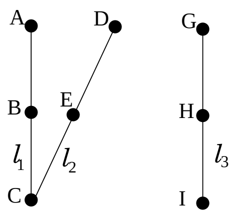


We verify that each axiom holds in this model:

$A1$ holds since ℓ_1 is incident with A, B, and C, ℓ_2 is incident with D, E, and F, and ℓ_3 is incident with G, H, and I.
$A2$ holds since there are exactly three lines: ℓ_1, ℓ_2, and ℓ_3.
$A3$ holds since no single point is incident with more then one line, so no pair of points can be incident with more than once line.
$A4$ holds since the lines in this model do not share any points.

(b) (6 points) Find a second model for this geometry that is not isomorphic to your previous model. Be sure to explain how you know that your second model is not isomorphic to your previous model.

Here is another possible model:



We again verify that each axiom holds in this model:

$A1$ holds since ℓ_1 is incident with A, B, and C, ℓ_2 is incident with D, E, and C, and ℓ_3 is incident with G, H, and I.
$A2$ holds since there are exactly three lines: ℓ_1, ℓ_2, and ℓ_3.
$A3$ holds since C is the only point that is incident with more then one line, so no pair of points can be incident with more than once line.
$A4$ holds since the lines ℓ_1 and ℓ_2 have the point C in common, and no other pairs of lines in this model share a point.

Notice that our first model had 9 points while the second has 8 points. Since any isomorphism would be a bijection that takes the set of points in one model to the set of points in the other model, there can is no bijection between two finite sets of different cardinalities, so these models are not isomorphic.

(c) (4 points) Write the duals of the axioms $A2$, and $A3$.

$A2'$: There are exactly three points.

$A3'$: Every pair of distinct lines has at most one point of concurrency.

(d) (6 points) Does this axiomatic system satisfy the principle of duality? Why or why not?

There are several ways to illustrate the fact that this system does not satisfy the principle of duality. Here is one of them. Notice that neither of the models above have 3 points. Since there are models for this axiomatic system that do not satisfy $A2'$, this dual axiom is not a true theorem in this system, so this axiomatic system does not satisfy the principle of duality.

4. Consider the following axiomatic system:

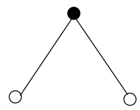
$A1$: Each **bot** *pats* exactly 2 **tobs**.

$A2$: For each pair of distinct **tobs**, there is a **bot** that *pats* both **tobs**.

$A3$: There are exactly 5 **tobs**

(a) (6 points) Pick one of the three axioms and prove that it is independent from the other axioms in the system. Be sure to justify your answer.

The easiest axiom to show independence for is probably axiom 3. Consider the following model. Here, black vertices represent **bots**, white vertices represent **tobs**, and there is an edge between a **bot** and a **tob** precisely when the **bot** *pats* that **tob.**



In our model, notice that there is one **bot** and two **tobs**. Moreover, the **bot** *pats* both **tobs**. Therefore, axioms $A1$ and $A2$ are satisfied by this model. However, since there are not 5 **tobs**, axiom $A3$ is not satisfied. Hence axiom $A3$ is independent.

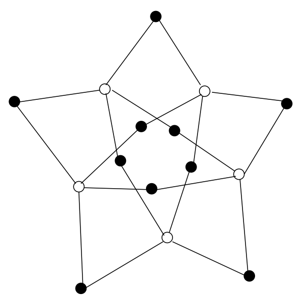
Note: All three of these axioms can be shown to be independent using appropriate models.

(b) (8 points) Find the minimum number of **bots** present in a model for this system. Be sure to justify your answer using the axioms given above.

Claim: There are at least 10 **bots**.

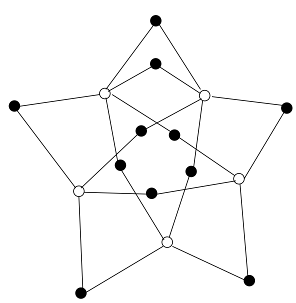
Proof: By Axiom $A3$, there are exactly 5 **tobs**. By Axiom $A2$, for each pair of distinct **tobs**, there is a **bot** that *pats* both **tobs**. Notice that there are $C(5,2) = 10$ distinct pairs of **tobs**. Finally, by Axiom $A1$, Each **bot** *pats* exactly 2 **tobs**. Thus no one **bot** can *pat* more than one pair of **tobs**. Hence, there must be at least one bot for each distinct pair of **tobs**. Therefore, there must be at least 10 **bots**.

The following model demonstrates that 10 **bots** is sufficient. Notice that there are exactly 5 **tobs**, each **bot** *pats* exactly two tobs, and that each of the 10 possible pairs if **tobs** is *patted* by a **bot**.



(c) (6 points) Is there a model with more **bots** than the minimum number you found above? Justify your answer.

A careful reading of axioms will show that although there must be a **bot** that *pats* any pair of distinct **tobs** and that every **bot** *pats* exactly 2 **tobs**, there is nothing that prevents more than one **bot** *patting* the same pair of **tobs**. Consider the following model:



This model has 11 **bots**. Since any **bot** can be duplicated any number of times, there is no upper bound on the number of **bots** in this axiomatic system.

Recall the Axioms for a Fano's Geometry:

Axiom 1: There exists at least one line.

Axiom 2: Every line has exactly three points incident to it.

Axiom 3: Not all points are incident to the same line.

Axiom 4: There is exactly one line incident with any two distinct points.

Axiom 5: There is at least one point incident with any two distinct lines.

5. (8 points) Prove the following Theorem:

   Each point in Fano's geometry is incident with exactly three lines. (Hint: Use two cases)

   Note: There are two main approaches that were used for this proof. The more difficult option was to carefully construct the entire model for Fano's Geometry and them observe that the resulting geometry satisfies the stated Theorem. The challenge is that you have to not only show that the standard model satisfies the axioms for the system, you must show that each step in the construction of the model is axiomatically justified rather than an optional choice. Those that take this approach tend to have great difficulty justifying the steps in their construction.

   We will take a more direct approach:

   Proof:

   Let P be a point in Fano's Geometry. By Axiom 1, there exists a line ℓ. We will consider two cases: the case that P is not on ℓ and the case that P is on ℓ (the order here is not accidental).

   Case 1: Assume P is not on line ℓ. By Axiom 2, there are three distinct points on the line ℓ, call them A, B, and C. Applying Axiom 4, there must be lines AP, BP, and CP. Note that these three lines must be distinct since if these lines were not distinct, this would contradict the uniqueness part of Axiom 4, since P would be a line with (at least) two of the points A, B, or C, and these points are already on ℓ. This shows that there are at least 3 lines incident with the point P.

   To see that there are exactly 3, suppose that there is a line k through P. Then by Axiom 5, there is a point D incident with both ℓ and k. By Axiom 2, D must be either A, B, or C. If $D = A$, then $k = DP = AP$. Similarly, for points B and C. Hence, the three lines AP, BP, and CP are the only lines incident with P.

   Case 2: Assume P is on line ℓ. Then by Axiom 2, there are exactly two other points incident with line ℓ. Call these points A and B. By Axiom 3, there must be a point Q not on line ℓ. Applying Case 1 to Q, PQ, AQ, and BQ are the only distinct lines incident with Q. Also note that P is not incident with line AQ. Hence, since P is not on line AQ, we may apply Case 1 to the point P and the line AQ to conclude that there are exactly three lines incident with P.

   Since these two cases exhaust all possibilities, we conclude that each point is incident with exactly three lines. □