Math 487

Chapter 1 Quiz - Practice Problem Solutions

1. Explain each of the following in your own words:
   (a) What is an axiomatic system?
   See course notes.
   (b) What is required for an axiomatic system to be consistent?
   See course notes.
   (c) What is required for an axiomatic system to be independent?
   See course notes.
   (d) What is required for an axiomatic system to be complete?
   See course notes.
   (e) What is required for an axiomatic system to satisfy the principle of duality?
   See course notes.

2. Is there a projective plane of order 1? If there is, how many points and lines are in the geometry? If not, why not?

   No. If there were a projective plane of order $n = 1$, then, by definition, it would have at least one line with exactly $n + 1 = 1 + 1 = 2$ points on it. This contradicts axiom $P3$, which states that every line has at least 3 points incident with it.

3. Is there a projective plane of order 11? If there is, how many points and lines are in the geometry? If not, why not?

   Yes. O. Veblen and W. Bussey proved that there are projective planes of order p^m for any prime number p and any integer m. By theorem $P5$, there are exactly $n^2 + n + 1$ points and $n^2 + n + 1$ lines in a projective plane of order n. Therefore, there are $11^2 + 11 + 1 = 121 + 11 + 1 = 133$ points and 133 lines in a projective plane of order 11.

4. Consider the following axiomatic system:

   $A1$: For any two distinct points, there is exactly one line incident with both points.

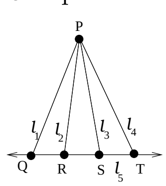
   $A2$: For any two distinct lines, there is at most one point incident with both lines.

   $A3$: Every line is incident with at least two points.

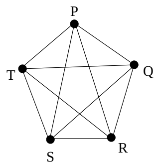
   $A4$: There are 5 points in this geometry.

   $A5$: No point is incident with every line.

   (a) State the undefined terms in this axiomatic system.
   The undefined or primitive terms in this axiomatic system are: *point, line, incident.*
   (b) Find a model for this geometry. Be sure to explain how you know that each axiom holds.
   One potential model for this geometry is:



   I will leave it to you to verify that this model satisfies the axioms.
   (c) Is this axiomatic system consistent? Justify your answer.
   Notice that since we are able to give a model for this axiomatic system that satisfies all of the axioms, this system must be consistent.
   (d) Find a second model for this geometry that is not isomorphic to your previous model. Be sure to explain how you know that each axiom holds and how you know this it is not isomorphic to your previous model.
   A second potential model for this geometry is:



I will leave it to you to verify that this model satisfies the axioms.

Notice that our first model had 5 points and 5 lines while our second model has 5 points and 10 lines. Since the cardinality of these sets are not the same, there cannot be a bijection between them, so these models cannot be isomorphic.

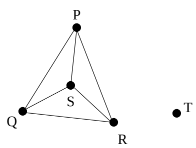
(e) Is this axiomatic system complete? Justify your answer.

This axiomatic system is not complete. Since there are two non-isomorphic models, there are statements about this axiomatic system that cannot be proved or disproved. For example, the statement: "There are exactly 5 lines in this geometry" cannot be proven and cannot be disproved.

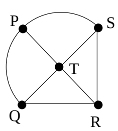
(f) Which of the axioms in this model are independent? Justify your answer.

I claim that the axioms $A1$, $A2$, $A3$, $A4$ and $A5$ are all independent.

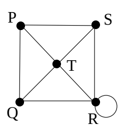
To see the independence of $A1$, I claim that the following model satisfies the axioms $A2$, $A3$, $A4$ and $A5$ but does not satisfy $A1$. You should verify this claim.



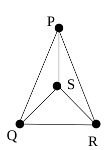
To see the independence of $A2$, I claim that the following model satisfies the axioms $A1$, $A3$, $A4$ and $A5$ but does not satisfy $A2$. You should verify this claim.



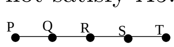
To see the independence of $A3$, I claim that the following model satisfies the axioms $A1$, $A2$, $A4$ and $A5$ but does not satisfy $A3$. You should verify this claim.



To see the independence of $A4$, I claim that the following model satisfies the axioms $A1$, $A2$, $A4$ and $A5$ but does not satisfy $A4$. You should verify this claim.



To see the independence of $A5$, I claim that the following model satisfies the axioms $A1$, $A2$, $A3$ and $A4$ but does not satisfy $A5$. You should verify this claim.



(g) Prove the theorem $T1$: There are at least five lines in this geometry.

By $A4$, there are 5 points in this geometry. Call them A, B, C, D, and E, respectively. By $A1$, there is exactly one line ℓ_1 that is incident with both A and B. By $A5$, there is at least one point that is not incident with ℓ_1. Without loss of generality, suppose that E is not incident with ℓ_1.

By $A1$, there are lines ℓ_2 and ℓ_3 incident with the pairs of points A, E and B, E respectively. Notice that ℓ_2 and ℓ_3 are distinct from ℓ_1 since E is not on ℓ_1. Also ℓ_2 and ℓ_3 are distinct, since if $\ell_2 = \ell_3$, there would be two lines incident with the pair of points A, B, which contradicts axiom $A2$.

Next, consider the point C.

Case 1: Suppose C is not incident with ℓ_1 or ℓ_2 or ℓ_3, then, by axiom $A1$, there must be lines ℓ_4, ℓ_5 ℓ_6 incident with the pairs of points A, C, B, C, and E, C respectively. As above, by axiom $A2$ these lines must all be distinct, so there are at least 5 lines in this geometry which concludes the proof in this case.

Case 2a: Suppose C is incident with ℓ_1. Then, by axiom $A2$, C is not incident with ℓ_2 or ℓ_3. Therefore, by axiom $A1$, there is exactly one line incident with the pair of points E, C. Call it ℓ_4. As above, by axiom $A2$ this line must be distinct from the previous lines.

Case 2b: Suppose C is incident with ℓ_2. Then, by axiom $A2$, C is not incident with ℓ_1 or ℓ_3. Therefore, by axiom $A1$, there is exactly one line incident with the pair of points B, C. Call it ℓ_4. As above, by axiom $A2$ this line must be distinct from the previous lines.

Case 2c: Suppose C is incident with ℓ_3. Then, by axiom $A2$, C is not incident with ℓ_1 or ℓ_2. Therefore, by axiom $A1$, there is exactly one line incident with the pair of points A, C. Call it ℓ_4. As above, by axiom $A2$ this line must be distinct from the previous lines.

In all of these cases, we have shown that there are four distinct lines in this geometry.

Now consider the point D.

Subcase (i) If D is not incident with ℓ_1, ℓ_2, ℓ_3, or ℓ_4, then, by axiom $A1$, there must be lines ℓ_5, ℓ_6 ℓ_7, ℓ_8 incident with the pairs of points A, D, B, D, and C, D, and E, D respectively. As above, by axiom $A2$ these lines must all be distinct, so there are at least 8 lines in this geometry which concludes the proof in this case.

Subcase (iia) Suppose D is incident with ℓ_1. Then, by axiom $A2$, D is not incident with ℓ_2, ℓ_3 or ℓ_4. Therefore, by axiom $A1$, there is exactly one line incident with the pair of points E, D. Call it ℓ_5. As above, by axiom $A2$ this line must be distinct from the previous lines, so there are at least 5 lines in this geometry.

Subcase (iib) Suppose D is incident with ℓ_2. Then, by axiom $A2$, D is not incident with ℓ_1, ℓ_3 or ℓ_4. Therefore, by axiom $A1$, there is exactly one line incident with the pair of points B, D. Call it ℓ_5. As above, by axiom $A2$ this line must be distinct from the previous lines, so there are at least 5 lines in this geometry.

Subcase (iic) Suppose D is incident with ℓ_3. Then, by axiom $A2$, D is not incident with ℓ_1, ℓ_2 or ℓ_4. Therefore, by axiom $A1$, there is exactly one line incident with the pair of points A, D. Call it ℓ_5. As above, by axiom $A2$ this line must be distinct from the previous lines, so there are at least 5 lines in this geometry.

Subcase (iid) Suppose D is incident with ℓ_4. Then, by axiom $A2$, D is not incident with ℓ_1, ℓ_2 or ℓ_3. Therefore, by axiom $A1$, there is exactly one line incident with the pair of points D and whichever point among A, B, E is not on ℓ_4. Call this line ℓ_5. As above, by axiom $A2$ this line must be distinct from the previous lines, so there are at least 5 lines in this geometry.

Since all possible cases result in there being at least 5 lines in the geometry, the statement is proven. □

(h) Write the duals to $A3$ and $A5$. Are these dual statements true statements in this axiomatic system?

$A3'$: Every point is incident with at least two lines.

Claim: the dual of $A3$ is true.

Proof: Let P be a point. Let P' be a point distinct from P ($A4$ guarantees the existence of this point). By $A1$, there is exactly one line ℓ_1 incident with P and P'. By $A5$, no point is incident with every line, so there must be at least one line ℓ_2 that is not incident with P. By $A2$, there is exactly one point incident with both ℓ_1 and ℓ_2. By A_3, there are at least two points incident with the line ℓ_2. In particular, there must be a point P'' not on ℓ_1 and incident with ℓ_2. Therefore, since P is not incident with ℓ_2 and P'' is not incident with ℓ_1, by $A1$, there must be a third line ℓ_3 that is incident with both P and P''. Hence there are two distinct lines incident with P. Since P was arbitrary, this proves the statement. □

$A5'$: No line is incident with every point.

Claim: the dual of $A5$ is true.

Proof: Suppose, to obtain the contradiction, that there is a line ℓ such that the points A, B, C, D, E are all incident with this line. By $A5$, there is at least one line that is not incident with point A. Call this line ℓ'. Notice that ℓ and ℓ' are distinct since A is incident with ℓ but not with ℓ'. By A_3, there are at least two points incident with the line ℓ', so there must be a second point P on ℓ'. However, since there are only 5 points in the geometry, P must be equal to one of B, C, D, E. However, this means that both ℓ and ℓ' are incident with the same pair of points, contradicting $A1$. This completed the proof of the statement. □

(i) Does this axiomatic system satisfy the principle of duality? Why or why not?

Notice that $A1$ and $A2$ are dual statements. Also, from part (h) above, the duals of $A3$ and $A5$ hold. What remains is to consider the dual of axiom $A4$, which is the statement: There are 5 lines in this geometry.

Recall that in part (d) above, we found a model for this geometry that has more than 5 lines. In fact, we discussed the fact that the statement: "There are 5 lines in this geometry" cannot be proved in this model. Therefore, this axiomatic system does not satisfy the principle of duality.