Math 487
Exam 2 **Name:**______________________________

**Instructions:** You will have 55 minutes to complete this exam. The credit given on each problem will be proportional to the amount of correct work shown. Answers without supporting work will receive little credit.

Work exam problems on separate sheets of paper (with the exception of problem #6).

1. (4 points each) True or False (give a **brief** justification of your answers):

   (a) Given three collinear points in a neutral geometry, one of them is between the other two.

   **True**. Consider three collinear points $A$, $B$, and $C$. Since these points are collinear, we can use the Ruler Placement postulate to assign coordinates to these three points in such a way that one is zero and another is positive. WLOG, assume that $A, B$ and $C$ and the ruler used assign coordinates $a, b$ and $c$ respectively so that $0 = a < b < c$. Then, using the Ruler Postulate, $d(A, B) = b$, $d(B, C) = c - b$, and $d(A, C) = c$. Then $d(A, B) + d(B, C) = b + (c - b) = c = d(A, C)$. Hence $A - B - C$.

   (b) If the sum of the measure of two angles is 180, then the two angles form a linear pair.

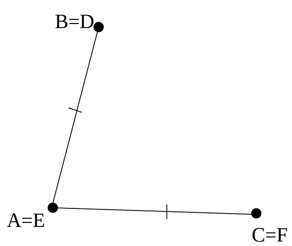
   **False**. Note that two angles whose measures sum to 180 need not share a vertex and a common ray. Therefore, they need not form a linear pair.

   (c) The statement "A rectangle exists" is equivalent to Euclid's 5th Postulate.

   **True**. Recall: **Euclidean Proposition 2.23** There exists a rectangle. As discussed in class, this Proposition is equivalent to the Euclidean Parallel Postulate, which is equivalent to Euclid's 5th Postulate.

   (d) If $\angle BAC = \angle DEF$, $\overline{AB} \cong \overline{DE}$ and $\overline{AC} \cong \overline{EF}$ then $B = D$, $A = E$, and $C = F$.

   **False**. It is tempting to think that this is true, since we do know from what is stated that $A = E$. However, the special case where $\overline{AB} \cong \overline{AC}$ provides a counterexample, as shown in the figure below.

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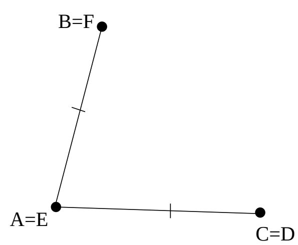



2. (6 points each) For each statement given below, state one geometric model where the given statement is **False** and **briefly** state how you know that it is false (you *do not* have to provide a specific counterexample).

   (a) The ruler postulate holds.

   The Ruler Postulate does not hold in Discrete Planes, in the Riemann Sphere, or in the Modified Riemann Sphere.

   For each of these the issue is since that there is a maximum distance between pairs of points in these models, any map between the points of a line and the set of real numbers that respects distance will not be onto.

   (b) The line uniqueness postulate holds.

   The Line Uniqueness Postulate does not hold in some Discrete Planes or in the Riemann Sphere (warning: it **does** hold in the Modified Riemann Sphere).

   In particular, if the Riemann Sphere, there are infinitely many lines between any pair of antipodal points.

(c) The plane separation axiom holds.

The Plane Separation Postulate does not hold in the Missing Strip Plane. You were asked to create a counterexample on one of your assigned homework problems. The way to accomplish this is to look at a pair of cartesian lines that intersect somewhere in the removed strip. This give a line that divides the plane into an edge and two half planes for which there is a pair of points on opposite sides, but the segment joining these points does not intersect the original line.

(d) The SAS Postulate Holds.

The SAS Postulate does not hold in the Taxicab plane or in the Max Distance Plane. Nearly all of you found a counterexample to this on an assigned homework problem. Simply referencing your counterexample sould suffice.

3. (12 points) Prove **one** of the following theorems:

$T1$ : The base angles of an isosceles triangle are congruent.

**Proof:**

Let $\triangle ABC$ be an isosceles triangle with $\overline{AB} \cong \overline{AC}$. Since every angle has a unique angle bisector, let $\overrightarrow{AD}$ be the bisector of $\angle BAC$. By the Crossbar Theorem, $\overrightarrow{AD}$ and $\overline{BC}$ intersect in a unique point $E$ satsfying $B - E - C$. By definition of an angle bisector, $\angle BAE = \angle BAD \cong \angle CAD = \angle CAE$. Using the reflexivitty of segment congruence, $\overline{AE} \cong \overline{AE}$. Thus, by the SAS Postulate, $\triangle ABE \cong \triangle ACE$. Then, by definition of triangle congruence, $\angle CB = \angle EBA \cong \angle ECA = \angle BCA$. Hence the base angles of an isosceles triangle are congruent. □

$T2$ : The Summit angles of a Saccheri Quadrilateral are congruent.

**Proof:**

Let $\square ABCD$ be a Saccheri Quadrilateral. Then $\overline{AD} \cong \overline{BC}$, and $m(\angle DAB) = m(\angle ABC) = 90$. Notice that $\overline{AB} \cong \overline{BA}$, since segment congruence is reflexive. Using Line Uniqueness, the distance postulate, and the definition of betweenness, we can construct the diagonal segments $\overline{AC}$ and $\overline{DB}$. Then by the SAS Postulate, $\triangle DAB \cong \triangle CBA$. Therefore, using the definition of triangle congruence, $\overline{AC} \cong \overline{DB}$.

Now consider $\triangle CAD$ and $\triangle DBC$. We already know that $\overline{AD} \cong \overline{BC}$ and $\overline{AC} \cong \overline{BD}$ from above. Moreover, $\overline{DC} \cong \overline{DC}$, since segment congruence is reflexive. Then, applying the SSS Theorem, $\triangle CAD \cong \triangle DBC$. Therefore, using the definition of triangle congruence, $\angle ACD \cong \angle BCD$. That is, the summit angles of a Saccheri Quadrilateral are congruent. □

4. (15 points) Prove **one** of the following theorems in a neutral geometry:

$T1$ : Any angle has a unique angle bisector.

**Proof:**

Let $\angle ABC$ be an angle and suppose that $m(\angle ABC) = r$ (this angle has some measure by the Angle Measure Postulate). Using the Angle Construction Postulate, there is a ray $\overrightarrow{BD}$ on the same side of $\overleftrightarrow{AB}$ as the point $C$ such that $m(ABD) = \frac{r}{2}$. Using Theorem 2.7, since $m(ABD) < m(\angle ABC)$ and $D$ is on the same side of $\overleftrightarrow{AB}$ as the point $C$, then $D \in int(\angle ABC)$. Therefore, using the Angle Addition Postulate, $m(ABD) + m(\angle DBC) = m(\angle ABC)$. Therefore, $m(ABD) = m(\angle DBC) = \frac{r}{2}$. Hence $\overrightarrow{BD}$ is an angle bisector of $\angle ABC$.

To see that $\overrightarrow{BD}$ is unique, suppose that there is a ray $\overrightarrow{BE}$ that bisects $\angle ABD$. Then $E \in int(\angle ABC)$ and satisfies $m(ABE) = m(\angle EBC)$. But since $m(\angle ABC) = r$, again using the Angle Addition Postulate, $m(ABE) + m(\angle EBC) = m(\angle ABC)$. Therefore, $m(ABE) = m(\angle EBC) = \frac{r}{2}$.

However, by the uniqueness portion of the Angle Construction Postulate, only one ray in the same half plane may have a given measure. Hence we must have $\overrightarrow{BD} = \overrightarrow{BE}$, hence the angle bisector of an angle is unique. □

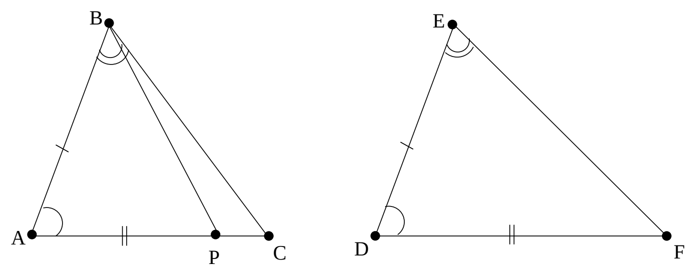
$T2$ : An isosceles triangle exists.

**Proof:**

By SMSG Postulate 5a (Existence of Points), there are three distinct non-collinear points. Call these $A$, $B$, and $C$. By SMSG Postulate 1 (Line Uniqueness), there is a line $\overleftrightarrow{AB}$ and a line $\overleftrightarrow{AC}$. Since $A, B$, and $C$ are non-collinear, these lines must be distinct. Using the Ruler Postulate, there is a point $D$ on the ray $\overrightarrow{AC}$ such that $d(A,B) = d(A,D)$. Again, using SMSG Postulate 1 (Line Uniqueness), there is a line $\overleftrightarrow{BD}$. Since $d(A,B) > 0$, $A \neq D$, so $\overleftrightarrow{BD}$ is distinct from $\overleftrightarrow{AB}$. Similarly, $\overleftrightarrow{BD}$ is distinct from $\overleftrightarrow{AC} = \overleftrightarrow{AD}$. Consider the triangle $\triangle ABD$. Since $d(A,B) = d(A,D)$, $\overline{AB} \cong \overline{AD}$. Hence this triangle is isosceles. $\square$

$T3$ : The ASA Theorem.

**Proof:**




Let $\triangle ABC$ and $\triangle DEF$ be triangles satisfying $\angle CAB \cong \angle FDE$, $\overline{AB} \cong \overline{DE}$, and $\angle ABC \cong \angle DEF$. To obtain a contradiction, suppose that $\overline{AC} \ncong \overline{DF}$. WLOG, assume $d(A,C) > d(D,F)$. By the ruler postulate, there is a point $P$ on the same side of $\overleftrightarrow{AB}$ as $C$ such that $P$ is on the line $\overleftrightarrow{AC}$ and $d(A,P) = d(D,F)$. Then $A - P - C$ and $\overline{AP} \cong \overline{DF}$. Therefore, by the SAS Postulate, $\triangle ABP \cong \triangle DEF$. But then, using the definition of triangle congruence, $\angle ABP \cong \angle DEF$. But we already have that $\angle ABC \cong \angle DEF$ and $P \neq C$. This gives two distinct angles one the same side of $\overleftrightarrow{AB}$ with vertex $B$ having the same measure. This is a contradiction of the uniqueness portion of the Angle Measurement Postulate. Hence $\overline{AC} \cong \overline{DF}$, so using the SAS Postulate, $\triangle ABC \cong \triangle DEF$ $\square$

5. (12 points) Recall that the standard ruler for a non vertical line in the Taxicab plane:
$f : \ell_{m,b} \to \mathbb{R}$ is given by $f(x,y) = (1 + |m|)\,x$.
Do **one** of the following:

(a) Show that this ruler function is one-to-one.

**Proof:**

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be points on a non-vertical line $\ell$ given by the equation $y = mx + b$ for some real numbers $m$ and $b$. Further suppose that $f(P) = f(Q)$. Then $(1 + |m|)\,x + 1 = (1 + |m|)\,x_2$. Since $(1 + |m|) > 0$, we must have $x_1 = x_2$. Finally, since $y_1 = mx_1 + b$, $y_2 = mx_2 + b$, and $x_1 = x_2$, then $y_1 = y_2$. Hence $P = Q$, and therefore, $f$ is a one-to-one function. $\square$

(b) Given the points $P(3,-1)$, $Q(5,7)$ Find a ruler such that $f(P) = 0$ and $f(Q) > 0$.

**Solution:**

Notice that the points $P$ and $Q$ lie on a non-vertical line in the Taxicab plane. The slope of the line containing these points is given by $m = \dfrac{7-(-1)}{5-3} = \dfrac{8}{2} = 4$. Then the standard ruler in the Taxicab Plane for this line is given by: $f(x,y) = (1 + |m|)\,x = 5x$.
To find our desired ruler, we shift the standard ruler 3 units to the left, giving $g(x) = 5(x-3) = 5x - 15$. Notice that $f(P) = 0$ and $f(Q) = 25 - 15 = 10$.

**Note:** Problem #6 is on the reverse side of this page. Please complete it in the space provided.

6. (24 points) Provide the reasons for each step in the following proof.

**Theorem:** Let $\triangle ABC$ be a triangle with external angle $\angle BCD$ (so $A - C - D$), then $m(\angle BCD) > m(ABC)$ and $m(\angle BCD) > m(BAC)$.

| | |
|---|---|
| 1. Let $M$ be the midpoint of $\overline{BC}$. | The Ruler Postulate and the definition of a midpoint (or the theorem we proved about the existence of a unique midpoint of any segment) |
| 2. Consider the ray $\overrightarrow{AM}$. | The Line Uniqueness Postulate, the Ruler Postulate, and the definition of a ray. |
| 3. Let $E$ be the point on $\overrightarrow{AM}$ with $A - M - E$ and $\overline{AM} \cong \overline{ME}$. | The Ruler Postulate and the definition of betweenness |
| 4. Notice that $\angle AMB \cong \angle CME$. | The Vertical Angle Thoerem |
| 5. Then $\triangle AMB \cong \triangle CME$. | The SAS Postulate |
| 6. Hence $\angle ABC = \angle ABM \cong \angle MCE = \angle BCE$. | The definition of triangle congruence |
| 7. Since $A - M - E$, then $E \in int(\angle BCD)$. | The Plane Separation Postulate and the definition of the interior of an angle. ($E$ is on the same side of $\overleftrightarrow{BC}$ as $D$ and on the same side of $\overleftrightarrow{AD}$ as $B$) |
| 8. Then $m(\angle BCE) < m(\angle BCD)$. | Theorem 2.7 |
| 9. But then $m(\angle BCE) = m(\angle ABC) < m(\angle BCD)$. | This follows from steps 6 and 8, along with the properties of equality. |
| 10. Next, let $D'$ be chosen so that $B - C - D'$. | The Ruler Postulate and the definition of betweenness. |
| 11. Then $m(\angle BCD) = m(\angle ACD')$. | The Vertical Angle Theorem |
| 12. By the same argument as above, $m(\angle ACD') > m(\angle BAC)$. | repeat steps 1 through 8 above, relabeling as needed. |
| 13. Hence $m(\angle BCD) > m(\angle BAC)$. This completes the proof. | This follows from steps 11 and 12, along with the properties of equality. |