Math 487
Exam 3

Instructions: You will have 55 minutes to complete this exam. The credit given on each problem will be proportional to the amount of correct work shown. Answers without supporting work will receive little credit.

Work your exam on separate sheets of paper. Be sure to put your name on at least the front page.

Note: Throughout this exam, you may assume that both function composition and matrix multiplication are associative.

1. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (x^3, 5y)$.

   (a) (6 points) Is this function a transformation? Justify your answer.

   Yes. We will show that f is both 1-1 and onto. To see that f is 1-1, suppose that there are inputs (a, b) and (c, d) such that $f(a, b) = f(c, d)$. Then $(a^3, 5b) = (c^3, 5d)$. Therefore, $a^3 = c^3$, so $a = c$, and $5b = 5d$, so $b = d$. Hence f is 1-1.

   To see that f is onto, let (a, b) be a point in $\mathbb{R}^2$, let $x = \sqrt[3]{a}$ and $y = \frac{b}{5}$. Note that $(x, y) \in \mathbb{R}^2$. Then $f(x, y) = \left((\sqrt[3]{a})^3, 5 \cdot \frac{b}{5} \right) = (a, b)$. Hence f is onto and this f is a transformation.

   (b) (6 points) Is this function an isometry? Justify your answer.

   No. Consider the points $P(0, 0)$ and $Q(0, 1)$. Then $f(P) = f((0, 0)) = (0, 0)$ and $f(Q) = f((0, 1)) = (0, 5)$. Notice that $d(P, Q) = \sqrt{(1 - 0)^2 + (0 - 0)^2} = 1$, while $d(f(P), f(Q)) = \sqrt{(0 - 0)^2 + (0 - 5)^2} = 5$. Since $d(P, Q) \neq d(f(P), f(Q))$, f is not an isometry.

2. Consider the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

   (a) (5 points) Does this matrix represent an affine transformation? Justify your answer.

   Yes. Recall that in section 3.2.3, we proved that any matrix of the form $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix}$ is a transformation. Since the matrix above has this form, it must be a transformation.

   (b) (5 points) Does this matrix represent an isometry? Justify your answer.

   No. Recall that in Proposition 3.7 we proved that isometries have matrices of the form $\begin{bmatrix} \cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} \cos\theta & \sin\theta & a \\ \sin\theta & -\cos\theta & b \\ 0 & 0 & 1 \end{bmatrix}$.
   Then this transformation cannot be an isometry since the matrix above has the wrong form (in particular, $|a_{12}| \neq |a_{21}|$.)

   (c) (5 points) Find a point P such that A maps P to the origin in the plane.

   Suppose $\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.
   Then $x_1 + 2 = 0$, so $x_1 = -2$, and $x_1 + x_2 + 3 = 0$, or, substituting, $-1 + x_2 = -3$, so $x_2 = -2$. Therefore, the point $(-2, -1, 1)$ is mapped onto the origin by A (this is easily verified using matrix multiplication).

3. (6 points) Suppose f is an isometry of $\mathbb{E}^2$ that has more than one fixed point. What type(s) of isometry could f be? Justify your answer.

   Recall that if f were a non-trivial translation, then f would not have any fixed points. Similarly, if f were a glide reflection, then it would not have any fixed points. If f were a rotation, then its only fixed point would be the center of the rotation. Therefore, f cannot be a translation, a glide reflection, or a rotation.

   Finally, note that if f were a reflection, the every point on the line used to define the reflection is a fixed point, so f could be a reflection. Since every isometry of $\mathbb{E}^2$ must be one of these four types, f must be a reflection.

4. (4 points each)

   (a) Find homogeneous coordinates for the line containing the points $(3,-2,1)$ and $(-1,4,1)$

   Recall that one nice way of finding a line through two points is two put an arbitrary point along with the two given points and then use the determinant as follows:

$$det\begin{vmatrix} x & 3 & -1 \\ y & -2 & 4 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

   Then $x(-2-4) - 3(y-4) + (-1)(y+2) = 0$, so $-6x - 3y + 12 - y - 2 = 0$, or $-6x - 4y + 10 = 0$. Reducing this gives $3x + 2y - 5 = 0$

   Then one way of writing the homogeneous coordinates of the line containing these points is: $[3\,,2\,,-5]$ (of course any multiple of this is also correct).

   (b) Find the point of intersection of the lines $[3\,,-2\,,5]$ and $[4\,,1\,,3]$

   Two find the point of intersection of two lines, we can use a similar technique making use of the determinant:

$$det\begin{vmatrix} \ell_1 & \ell_2 & \ell_3 \\ 3 & -2 & 5 \\ 4 & 1 & 3 \end{vmatrix} = 0$$

   Then $\ell_1(-6-5) - \ell_2(9-20) + \ell_3(3+8) = 0$, so $-11\ell_1 + 11\ell_2 + 11\ell_3 = 0$, or $-\ell_1 + \ell_2 + \ell_3 = 0$. Notice that we scaled so that the third entry is $+1$.

   Then the point of intersection is $(-1,1,1)$.

5. Consider the points $P(3,0,1)$, $Q(0,2,1)$, and $R(1,0,1)$. Note that, if necessary, you may provide a matrix representation that is given a product of several matrices (that is, you do not need to perform matrix multiplication to find a single matrix representing the transformation).

   (a) (6 points) Find a matrix representation of a translation that maps the point P to the point Q.

   Notice that $\overrightarrow{PQ} = \langle -3, 2\rangle$. Therefore, $T_{PQ} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ represents a translation mapping P to Q.

   (b) (6 points) Find the matrix representation of a rotation that maps P to Q.

   There are several ways to do this, but the most straightforward is to use a rotation of 180° about the midpoint of the points P and Q. Notice that $M = (\frac{3}{2}, 1)$, so we will use the rotation $R_{M,180^\circ}$. To find a product of matrices representing this rotation, we recall that $R_{M,180^\circ} = T_{OM} \circ R_{O,180^\circ} \circ T_{MO}$.

$$\text{Then } R_{M,180^\circ} = \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 180^\circ & -\sin 180^\circ & 0 \\ \sin 180^\circ & \cos 180^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) (10 points) Find the matrix representation of a rotation that maps P to R.

There are several ways to do this, but the most straightforward is to let $N = (2, 0)$ and to use the rotation $R_{N,180^\circ}$. To find a product of matrices representing this rotation, we recall that $R_{N,180^\circ} = T_{ON} \circ R_{O,180^\circ} \circ T_{NO}$.

$$\text{Then } R_{N,180^\circ} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 180^\circ & -\sin 180^\circ & 0 \\ \sin 180^\circ & \cos 180^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(d) (10 points) Find the matrix representation of a reflection that maps P to Q.

First, we need to find the axis of reflection. This must be the line passing through the midpoint of the segment $\overline{PQ}$ and perpendicular to this segment.

Straightforward computations show that the axis of reflection is $\ell = [6 \; -4 \; -5]$.

From this, we compute the angle between this line and the x-axis, $h = [0\,1\,0]$ using the computational technique defined in class:

$$\tan(\theta) = \frac{p_1q_2 - p_2q_1}{p_1q_1 + p_2q_2} = \frac{(0)(-4) - (1)(6)}{(0)(6) + (1)(-4)} = \frac{3}{2}$$

Therefore, $\theta = \tan^{-1}\left(\frac{3}{2}\right)$. Notice that constructing the associated right triangle gives $\sin(\theta) = \frac{3}{\sqrt{13}}$ and $\cos(\theta) = \frac{2}{\sqrt{13}}$.

Now we are ready to build the matrices for this transformation. Let S be the point where ℓ meets the x-axis (some straightforward algebra shows that $S = (0, \frac{5}{6})$). Our strategy (although there are other correct strategies) will be to first use $R_{S,-\theta}$ to rotate ℓ onto h. Next, we will reflect across h. Finally, we will use $R_{S,\theta}$ to rotate h back to ℓ.

Using the same methods described in parts (b) and (c) above:

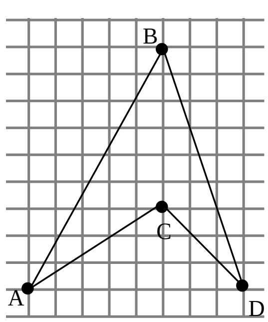
$$R_{S,-\theta} = T_{OS} \circ R_{O,-\theta} \circ T_{SO} = \begin{bmatrix} 1 & 0 & \frac{5}{6} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} & 0 \\ -\frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -\frac{5}{6} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$R_{S,\theta} = T_{OS} \circ R_{O,\theta} \circ T_{SO} = \begin{bmatrix} 1 & 0 & \frac{5}{6} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{13}} & 0 \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -\frac{5}{6} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

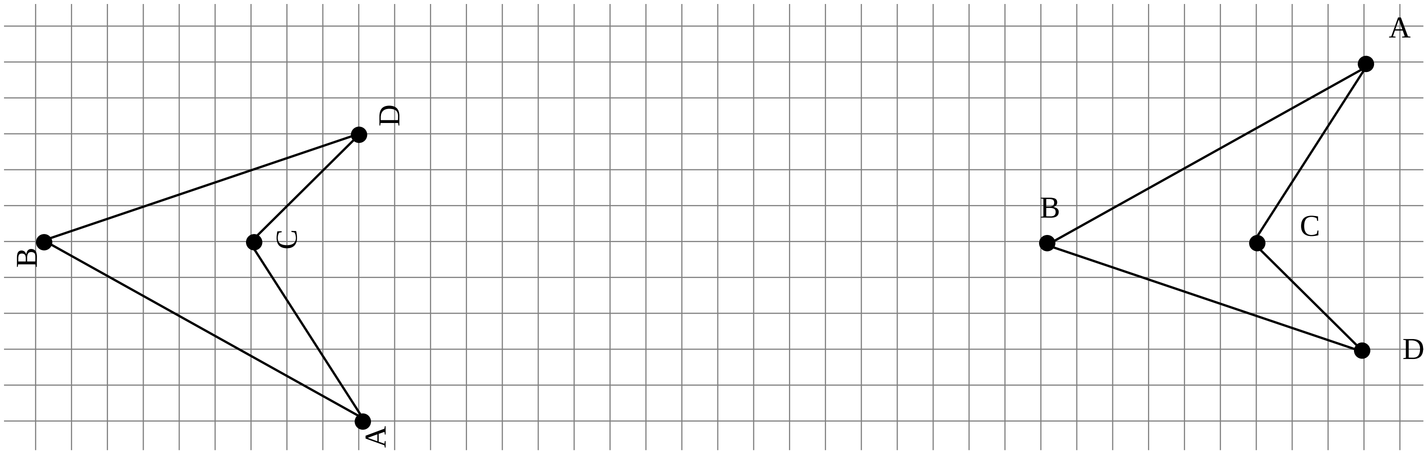
Hence $R_\ell = T_{OS} \circ R_{O,\theta} \circ T_{SO} \circ R_h \circ T_{OS} \circ R_{O,-\theta} \circ T_{SO}$

$$= \begin{bmatrix} 1 & 0 & \frac{5}{6} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{13}} & 0 \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -\frac{5}{6} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{5}{6} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} & 0 \\ -\frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -\frac{5}{6} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

6. (5 points each) Given the following planar figure, accurately draw its image under each of the following transformations.



(a) $R_{C,90^\circ}$ (b) R_ℓ where $\ell = \overleftrightarrow{CD}$



7. (12 points each) Prove **one** of the following:

   (a) The composition of two transformations is a transformation.

   Let f and g be transformations and consider $f \circ g$. To see that $f \circ g$ is 1-1, suppose that $f \circ g(P) = f \circ g(Q)$. That is, $f(g(P)) = f(g(Q))$. Since f is 1-1, we must have $g(P) = g(Q)$. But g is also 1-1, so we must have $P = Q$. Hence $f \circ g$ is 1-1.

   To see that $f \circ g$ is onto, let B be an element in the codomain of f. Then, since f is onto, there must be a C in the domain of f so that $f(C) = B$. Since g is onto, there must be a D in the domain of g so that $g(D) = C$. Then $f(g(D)) = f(C) = B$. Hence $f \circ g$ is onto.

   Since $f \circ g$ is both 1-1 and onto, it is a transformation.

   (b) The inverse of an isometry is an isometry.

   Let f be an isometry from D to R. By definition, f is a function that is both one-to-one and onto. Then we may define the inverse function f^{-1} as follows. For every $r \in R$, since f is 1-1 and onto, there is exactly one $d \in D$ such that $f(d) = r$. Then we define $f^{-1}(r) = d$. Then f^{-1} is a function. Since f is well defined, f^{-1} is onto (for every $d \in D$, there is an $r \in R$ such that $f(r) = d$). Also, f^{-1} is well defined since we for each $r \in R$, there is a $d \in D$ so that $f^{-1}(r) = d$ and there is exactly one such d since f is 1-1. Finally, f^{-1} must be 1-1 since if $f^{-1}(r_1) = f^{-1}(r_2) = d$, $f \circ f^{-1}(r_1) = f \circ f^{-1}(r_2) = f(d)$, so $r_1 = r_2$. Hence f^{-1} is a transformation.

   It remains to prove that the inverse of an isometry preserves the distance between any pair of points. Let f be an isometry from D to R. Let $r_1, r_2 \in R$ and suppose that $d(r_1, r_2) = M$. To obtain a contradiction, suppose that $d(f^{-1}(r_1), f^{-1}(r_2)) = M' \neq M$. Then, since f is an isometry, $M' = d\left(f\left(f^{-1}(r_1)\right), f\left(f^{-1}(r_2)\right)\right) = d(r_1, r_2) = M$. This is a contradiction. Hence f^{-1} is an isometry.

8. (12 points each) Prove **one** of the following:

   (a) Given a pair of distinct points X and Y in $\mathbb{E}$, there is a reflection that maps X to Y.

   Let X and Y be distinct points in $\mathbb{E}$. Consider $\overline{XY}$. Let M be the midpoint of $\overline{XY}$ [We proved on an earlier homework problem that there is a unique point M that is the midpoint of this segment. this was a consequence of the ruler postulate and the definition of betweenness]. Let ℓ be the line perpendicular to $\overline{XY}$ through the point M [This line must exist by the protractor postulate - construct the ray $\overrightarrow{MP}$ in one half plane of $\overleftrightarrow{XY}$ such that $m(\angle XMP) = 90$, then extend this to a line $\overleftrightarrow{MP}$]. If we take $\ell = \overleftrightarrow{MP}$, then R_ℓ is a reflection such that $R_\ell(X) = Y$. [notice that $\overline{XY}$ is perpendicular to $\overleftrightarrow{MP}$ and $\overline{XM} \cong \overline{MY}$].

   (b) The set of all translations of $\mathbb{E}^2$ is a group.

   - closure: Let T_1 and T_2 be two translations. Then T_1 and T_2 are represented by matrices of the form: $$T_1 = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \text{ and } T_2 = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that $T_1 \circ T_2 = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a+c \\ 0 & 1 & b+d \\ 0 & 0 & 1 \end{bmatrix}$.
Hence the set of all translations is closed under composition (matrix multiplication of the associated matrices).

- identity:

  The translation T_v by the vector $\vec{v} = \vec{0}$ is the identity transformation $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

- inverses:

  Given a translation $T_v = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$, $T_v^{-1} = \begin{bmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix}$. Notice that $T_v \circ T_v^{-1} = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix} =$
  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

- associativity:

  As noted at the beginning of this exam, I allowed you to assume that matrix multiplication is associative. Therefore, associativity is inherited from this property of matrices since we find the composition of transformations by multiplying their matrix representations.