Math 127 - Exam 1 **Name:**____________________
Version 1
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You MUST show appropriate work to receive credit

1. (3 points each) True or False (Include a *brief* justification of your answer):

   (a) $\sqrt{a^2+b^2} = a+b$

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   False.

   Notice that if $a = 2$ and $b = 3$, then $\sqrt{a^2+b^2} = \sqrt{4+9} = \sqrt{13}$ while $a+b = 2+3 = 5$.

   (b) If $x(x-1) = 3$, then $x = 3$, or $x = 4$.

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   False. The difficulty here is that since the right hand side of this equation is not zero, we cannot use the zero factor property. If we try to check these solutions, notice that $3(3-1) = 3(2) = 6$ and $4(4-1) = 4(3) = 12$, so neither of these values are in the solution set of this equation.

2. (5 points) Simplify the expression: $(2x+3y)^2 - (2x-3y)^2$

$(2x+3y)^2 - (2x-3y)^2 = (4x^2+6xy+6xy+9y^2) - (4x^2-6xy-6xy+9y^2)$

$= 4x^2+12xy+25y^2 - (4x^2-12xy+25y^2)$

$= 4x^2+12xy+25y^2 - 4x^2+12xy-25y^2$

$= 24xy.$

3. (5 points each) Factor each of the following *completely*. Box your answers.

   (a) $12x^2+5x-3$

   Using the "ac-split", notice that
   $ac = (12)(-3) = -36 = (9)(-4)$, and $9 + (-4) = 5$.
   Then $12x^2+5x-3 = 12x^2+9x-4x-4$
   $= 3x(4x+3) - 1(4x+3)$
   $= (4x+3)(3x-1)$.

   (b) $x^3+3x^2-4x-12$

   Using factoring by grouping,
   $x^3+3x^2-4x-12 = x^2(x+3) - 4(x+3)$
   $= (x+3)(x^2-4) = (x+3)(x+2)(x-2)$.

   (c) x^3-27

   Notice that this is a difference of cubes: (x^3-3^3).
   Therefore, using the special formula for a difference of cubes:

   $x^3-27 = (x-3)(x^2+3x+9)$

   (d) x^4-1

   Notice that this is a difference of squares,
   so $x^4-1 = (x^2+1)(x^2-1)$.

   The second term is also a difference of squares,
   so this factors further as $(x^2+1)(x+1)(x-1)$.

4. (7 points each) Perform the operations indicated and simplify each of the following as much as possible. Your answer should be completely reduced and should contain no complex fractions.

   (a) $\dfrac{x^2 - x - 12}{x^2 - 4} \div \dfrac{x^2 - 2x - 8}{x^2 + x - 6}$

   Factoring, we have $\dfrac{(x+3)(x-4)}{(x+2)(x-2)} \div \dfrac{(x-4)(x+2)}{(x+3)(x-2)}$

   Next, we change from multiplication to division: $\dfrac{(x+3)(x-4)}{(x+2)(x-2)} \cdot \dfrac{(x+3)(x-2)}{(x-4)(x+2)}$

   Now, we divide out common factors and combine into a single fraction: $\dfrac{(x+3)(x+3)}{(x+2)(x+2)}$.

   Thus our final simplified answer is: $\dfrac{(x+3)^2}{(x+2)^2}$

   (b) $\dfrac{8x + 12}{x^2 + 3x - 10} + \dfrac{x+1}{x+5}$

   First, we factor in order to obtain: $\dfrac{8x+12}{(x+5)(x-2)} + \dfrac{x+1}{x+5}$. Notice that the LCD is: $(x+5)(x-2)$

   Multiplying to get all terms over the LCD gives:

   $$\frac{8x+12}{(x+5)(x-2)} + \frac{x+1}{x+5} \cdot \frac{(x-2)}{(x-2)} = \frac{8x+12+(x+1)(x-2)}{(x+5)(x-2)}.$$

   Simplifying, we get:

   $$\frac{8x+12+x^2-x-2}{(x+5)(x-2)} = \frac{x^2+7x+10}{(x+5)(x-2)}$$

   $$= \frac{(x+5)(x+2)}{(x+5)(x-2)} = \frac{x+2}{x-2}.$$

5. (6 points) Use completing the square to solve the quadratic equation $2x^2 - 6x - 5 = 0$.

$2x^2 - 6x = 5$ [move the constant to the right hand side]

$x^2 - 3x = \frac{5}{2}$ [divide both sides by 2]

$x^2 - 3x + \left(\frac{5}{2}\right)^2 = \frac{5}{2} + \left(\frac{3}{2}\right)^2$ [add the appropriate constant]

$(x - \frac{3}{2})^2 = \frac{5}{2} + \frac{9}{4}$ [factor the left hand side]

$(x - \frac{3}{2})^2 = \frac{10}{4} + \frac{9}{4}$

$(x - \frac{3}{2})^2 = \frac{19}{4}$ [combine constants on the right hand side]

$x - \dfrac{3}{2} = \pm\sqrt{\dfrac{19}{4}}$ [take the square root of both sides]

$x = \dfrac{3}{2} \pm \dfrac{\sqrt{19}}{\sqrt{4}}$ [simplify]

$x = \dfrac{3}{2} \pm \dfrac{\sqrt{19}}{2}$

6. (4 points each) Perform the indicated operations and express your answer in the form $a + bi$:

   (a) $(3 - 5i)^2$

   $(3-5i)^2 = (3-5i)(3-5i) =$
   $9 - 15i - 15i + 25i^2 = 9 - 30i - 25$
   $= -16 - 30i$

   (b) $\dfrac{5 - 3i}{7 + 2i}$

   $\frac{5-3i}{7+2i} \cdot \frac{7-2i}{7-2i} = \frac{35-21i-10i+6i^2}{49+14i-14i-4i^2}$

   $= \frac{35-6-31i}{49+4} = \frac{29-31i}{53}$

   $= \frac{29}{53} + \frac{-31}{53}i$

   (c) $i^7\left(3i - i^3\right)$

   $= 3i^8 - i^{10} = 3(1) - (-1)$
   $= 3 + 1 = 4 = 4 + 0i$

   .

7. Find **all** solutions to the following equations:

(a) (5 points) $\frac{3}{5} + \frac{x}{2} = \frac{4-x}{7}$

We multiply by the LCD, $70 = (5)(2)(7)$: $(5)(2)(7) \cdot \left[\frac{7}{10} - \frac{x}{3} = \frac{2x-5}{2} \right]$

$(7)(2)(3) + (5)(7)x = (5)(2)(4-x)$, or $42 + 35x = 40 - 10x$.

Then $45x = -2$, so $x = -\frac{2}{45}$.

(b) (5 points) $\frac{3x-1}{6x-2} = \frac{2x+5}{4x+3}$

We multiply by the LCD, $(6x-2)(4x+3)$: $[(6x-2)(4x+3)]\frac{3x-1}{6x-2} = [(6x-2)(4x+3)]\frac{2x+5}{4x+3}$

$(4x+3)(3x-1) = (6x-2)(2x+5)$, or $12x^2 + 9x - 4x - 3 = 12x^2 - 4x + 30x - 10$

then $12x^2 + 5x - 3 = 12x^2 + 26x - 10$, so, combining terms (the x^2 term cancels), we get $7 = 21x$, thus $x = \frac{7}{21} = \frac{1}{3}$.

However, note that the value $x = \frac{1}{3}$ makes the term $6x - 2 = 6\left(\frac{1}{3}\right) - 2 = 2 - 2 = 0$, and division by zero is undefined. Therefore this solution does not check, so there is no solution.

Also note that this problem could be solved more easily and directly by noting that $\frac{3x-1}{6x-2} = \frac{3x-1}{2(3x-1)} = \frac{1}{2}$.

(c) (5 points) $5x^2 - x = -1$

We immediately rearrange the terms to put this quadratic into standard form: $5x^2 - x + 1 = 0$

Since this quadratic equation does not factor, the most straightforward way to solve this is to use the quadratic formula.

Notice that $a = 5$, $b = -1$, and $c = 1$.

Then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1 - 4(5)(1)}}{2(5)} = \frac{1 \pm \sqrt{1-20}}{10}$

So $x = \frac{1 \pm \sqrt{-19}}{10} = \frac{1 \pm i\sqrt{19}}{10} = \frac{1}{10} \pm \frac{i\sqrt{19}}{10}$

(d) (5 points) $x^3 - 3x^2 = 4x - 12$

We again begin by rearranging the terms to get everything on the same side: $x^3 - 3x^2 - 4x + 12 = 0$

Notice that this polynomial factors by grouping: $x^2(x-3) - 4(x-3) = 0$, or $(x-3)(x^2-4) = 0$

Using the zero product property to split into 2 cases, we see that either $x - 3 = 0$, so $x = 3$
or $x^2 - 4 = 0$, in which case, $x^2 = 4$, so $x = \pm\sqrt{4} = \pm 2$.
Then $x = 3$ or $x = 2$ or $x = -2$.

(e) (7 points) $\sqrt{3x+1} - \sqrt{x+4} = 1$

This problem ends up being a bit nicer if we rearrange to get: $\sqrt{3x+1} = 1 + \sqrt{x+4}$

Squaring both sides, we get: $\left(\sqrt{3x+1}\right)^2 = \left(1 + \sqrt{x+4}\right)^2$

which simplifies to give: $3x + 1 = 1 + 2\sqrt{x+4} + x + 4$ or $3x + 1 = x + 5 + \sqrt{x+4}$

Moving terms to isolate the remaining radical gives: $2x - 4 = 2\sqrt{x+4}$, or $x - 2 = \sqrt{x+4}$

Squaring again gives: $(x-2)^2 = x + 4$, or $x^2 - 4x + 4 = x + 4$

Then $x^2 - 5x = 0$, or $x(x-5) = 0$, so $x = 0$ or $x = 5$.

It is imperative that we check our answer, since the method of squaring can introduce extraneous solutions.

If $x = 0$, then $\sqrt{3x+1} - \sqrt{x+4} = \sqrt{1} - \sqrt{4} = 1 - 2 = -1$, so this solution does not check.

If $x = 5$, then $\sqrt{3x+1} - \sqrt{x+4} = \sqrt{16} - \sqrt{9} = 4 - 3 = 1$, so this solution does check.

Hence this equation has one solution: $x = 5$

(f) (7 points) $t^4 - t^2 - 12 = 0$

Here, we can either factor this expression, or, to make it a bit easier, we can substitute using $u = t^2$ and $u^2 = t^4$.

This gives $u^2 - u - 12 = 0$, or $(u-4)(u+3) = 0$

Therefore, either $u = 4$ or $u = -3$

That is, either $t^2 = 4$ or $t^2 = -3$

But then either $t = \pm\sqrt{2}$ or $t = \pm\sqrt{-3} = \pm i\sqrt{3}$.

Notice that all 4 of these solutions check.

8. (6 points) Fred and Wilma start from the same point in Bedrock and travel on a straight road. Fred travels at 30 mph, while Wilma travels at 50 mph. If Wilma starts 3 hours after Fred, find the distance they travel before Wilma catches up to Fred. [You must use algebra and the complete problem solving process to get full credit on this problem.]

Since we are taking about simple motion, the basic equation is $d = rt$. We will start by setting up separate models for Fred and Wilma and then we will compare the two. Our main variables will be t, the total time Fred spent traveling, and d, the distance both Fred and Wilma travel.

For Fred, $d_1 = r_1 t_1$, where $r_1 = 30$ miles per hour, and $t_1 = t$, the total travel time for Fred.

For Wilma, $d_2 = r_2 t_2$, where $r_1 = 50$ miles per hour, and $t_2 = t - 3$ (since Wilma started out 3 hours after Fred, her total travel time will end up being 3 hours less than Fred's).

Notice that since we are interested in the time when Wilma caught up to Fred, $d_1 = d_2 = d$.

Then we have the equation $r_1 t_1 = r_2 t_2$, or $30t = 50(t-3)$.
Therefore, $30t = 50t - 150$, or $150 = 20t$.
Then $t = \frac{150}{20} = 7.5$ hours. That is, it will take Wilma 7.5 hours to catch up with Fred.

Therefore, Fred and Wilma will have traveled $30(7.5) = 225$ miles ($50(4.5) = 225$ miles) when Wilma catches up with Fred.

Extra Credit: (5 points) Use completing the square on the general quadratic polynomial $ax^2 + bx + c = 0$ to derive the quadratic formula (You should write your supporting work on the back of this page).

$ax^2 + bx + c = 0$

$ax^2 + bx = -c$

$x^2 + \frac{b}{a}x = -\frac{c}{a}$

$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$

$(x + \frac{b}{2a})^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$

$x + \frac{b}{2a} = \pm\sqrt{-\frac{c}{a} + \frac{b^2}{4a^2}} = \pm\sqrt{\frac{-4ac}{4a^2} + \frac{b^2}{4a^2}} = \pm\sqrt{\frac{-4ac+b^2}{4a^2}}$

$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$