Math 127 - Exam 4 **Name:**________________________________
Version 1
05/01/2015

You MUST show appropriate work to receive credit

1. Suppose $f(x) = 2 + 5x - 3x^2$

   (a) (5 points) Find the vertex and axis of symmetry for this quadratic function.

   First note that written in standard quadratic function form, $f(x) = -3x^2+5x+2$. We find the vertex algebraically as follows:
   $h = \dfrac{-b}{2a} = -\dfrac{5}{2(-3)} = \dfrac{5}{6}$. Then $f(h) = f\left(\frac{5}{6}\right) = -3\left(\frac{5}{6}\right)^2 + 5\left(\frac{5}{6}\right) + 2 == -\frac{25}{12} + \frac{25}{6} + 2 = -\frac{25}{12} + \frac{50}{12} + \frac{24}{12} = \frac{49}{12}$.
   So the vertex is $V : \left(\dfrac{5}{6}, \dfrac{49}{12}\right)$ and the axis of symmetry is $x = \dfrac{5}{6}$.

   (b) (6 points) Find all intercepts.

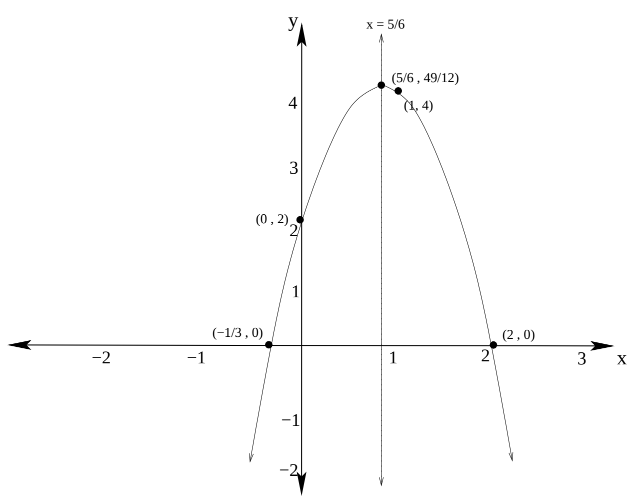
   First, we find the y-intercept: $f(0) = 2$, so $(0, 2)$ is the y-intercept.

   To find the x-intercepts, we factor to get $(3x + 1)(-x + 2) = 0$, which has solutions $x = -\frac{1}{3}$ and $x = 2$.
   Alternatively, one could use the quadratic formula to find these solutions.
   Then the x-intercepts are $\left(-\frac{1}{3}, 0\right)$ and $(2, 0)$.

   (c) (6 points) Graph this function on the axes provided.

   Finally, we plot a few more values to get a better idea of the shape of our graph: $f(1) = 2 + 5 - 3 = 4$, so $(1, 4)$ is on the graph, and $f(-1) = 2 - 5 - 3 = -6$, so $(-1, -6)$ is on the graph.

   Combining this with the information found above gives the following graph:



2. (8 points) Use polynomial long division to divide $4x^4 + 6x^3 + 3x - 1$ by $2x^2 + 1$.

$$
\begin{array}{r|l}
 & \quad\quad\quad\quad\; 2x^2 + 3x - 1 \\ \hline
2x^2 + 1 & 4x^4 + 6x^3 \qquad\quad + 3x - 1 \\
 & -4x^4 \qquad - 2x^2 \\ \hline
 & \qquad 6x^3 - 2x^2 + 3x \\
 & \quad -6x^3 \qquad - 3x \\ \hline
 & \qquad\quad -2x^2 \qquad - 1 \\
 & \qquad\quad\;\; 2x^2 \qquad + 1 \\ \hline
 & \qquad\qquad\qquad\quad 0
\end{array}
$$

3. Suppose a football is thrown down field by a quarterback. Its height above the ground is modeled by $y = -0.025x^2 + x + 6$, where y is the height of the football in feet and x is the horizontal distance of the ball from the quarterback, in yards.

(a) (4 points) From what height was the football thrown?

Notice that since x represents the distance between the ball and the Quarterback and the ball in yards, the ball is thrown when $x = 0$.

Since y represents the height of the ball, in feet, we need to find y when $x = 0$. Here, $y = -0.025(0)^2 + (0) + 6 = 6$.

Therefore, the ball is 6 feet above the ground when it is thrown.

(b) (6 points) What is the ball's maximum height (the height at the highest point that the ball reaches)?

First, notice that the maximum height must occur at the vertex. To find the vertex, we first find $h = \frac{-b}{2a} = \frac{-1}{2(-0.025)} = \frac{1}{0.05} = 20$.

To find the maximum height, we find y when $x = 20$: $y = -0.025(20)^2 + (20) + 6 = 16$

Therefore, the maximum height attained by the ball is 16 feet above the ground.

(c) (8 points) Assuming that the ball is overthrown and is not touched by any player, how from from where it was thrown does the football hit the ground?

We know that the ball hits the ground when $y = 0$. Then we are looking for positive solutions to the equation $0 = -0.025x^2 + x + 6$. Since this does not factor easily, it is probably best to use the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 - 4(6)(-0.025)}}{2(-0.025)} = \frac{-1 \pm \sqrt{1.6}}{-0.05}$$

This yields two possible solutions: $x \approx 45.298$ and $x =\approx -5.298$. Only the positive solution makes physical sense, so we conclude that the football hits the ground about 42.3 yards downfield from where it was thrown.

4. Let $f(x) = x^4 - 3x^3 + x^2 - 7$.

(a) (5 points) Use synthetic division to find $f(-2)$.

First note that we should think of this polynomial as $f(x) = x^4 - 3x^3 + x^2 + 0x - 7$, supplying the missing x term with a zero coefficient.

$$\begin{array}{r|rrrrr} & 1 & -3 & 1 & 0 & -7 \\ -2 & & -2 & 10 & -22 & 44 \\ \hline & 1 & -5 & 11 & -22 & 37 \end{array}$$

The synthetic division given above shows that $f(-2) = 37$.

(b) (5 points) Write $f(x)$ in the form $f(x) = (x + 2)q(x) + r$.

Interpreting the result of the synthetic division that we applied above, we see that
$f(x) = (x + 2)(x^3 - 5x^2 + 11x - 22) + 37$

5. Given the function $f(x) = -\frac{1}{2}(x+2)^2(x-1)^3(x+3)$.

   (a) (6 points) Find the leading term of $f(x)$ and use it to determine the end behavior of the graph of $f(x)$.

   One way to find the leading term is to multiply out the entire polynomial and gather the terms. This is a lot more work than is necessary as we only want the first term, not the entire polynomial. To find the leading term, we combine the coeficients and highest order terms from each factor:

   Then we have $-\frac{1}{2} \cdot x^2 \cdot x^3 \cdot x = -\frac{1}{2}x^6$.

   Since our leading term has a negative coefficient and even degree, both the left and right ends of the graph extend down toward $-\infty$ in the y direction.

   (b) (6 points) Find the zeros of $f(x)$ and their multiplicities.

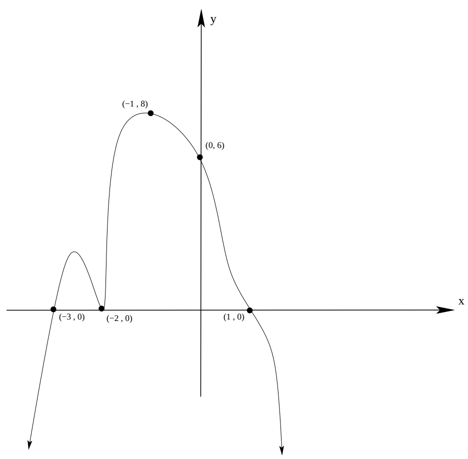
   We find the zeros by looking at each factor of $f(x)$. Note that the coeficient $-\frac{1}{2}$ does not correspond to a zero.

   The first factor is zero when $x = -2$. This root has multiplicity 2, so the graph has a turning point there.

   The second factor is zero when $x = 1$. This root has multiplicity 3, so the graph crosses the x-axis there.

   The third factor is zero when $x = -3$. This root has multiplicity 1, so the graph crosses the x-axis there.

   (c) (8 points) Graph the function $f(x)$ on the axes provided.



6. Let $f(x) = 6x^4 - 2x^2 + 5$.

   (a) (5 points) List all possible rational zeros of this function.

   $a_0 = 5$, so possible values for p are: $p = \pm 1, \pm 5$

   $a_n = 6$ so possible values for q are: $q = \pm 1, \pm 2, \pm 3, \pm 6$.

   Then list of all possible rational zeros is given by: $\frac{p}{q} = \pm 1, \pm 5, \pm\frac{1}{2}, \pm\frac{5}{2}, \pm\frac{1}{3}, \pm\frac{5}{3}, \pm\frac{1}{6}, \pm\frac{5}{6}$

   (b) (5 points) Choose one possible rational zero and use synthetic division to check to see whether or not it is a zero of $f(x)$.

   There are several possible answers to this question. All you needed to do was to choose one of the (correct) possible rational zeros that you found in part (a) and run through the algorithm. I was surprised at how many of you chose 5 or one of the fractions. Here is one of the simpler options:

$$\begin{array}{r|rrrrr} & 6 & 0 & -2 & 0 & 5 \\ 1 & & 6 & 6 & 4 & 4 \\ \hline & 6 & 6 & 4 & 4 & 9 \end{array}$$

   The synthetic division given above shows that $f(1) = 9$. Since the result is non-zero, we know that $x = 1$ is not a zero of $f(x)$.

(c) (5 points) Use Descartes' Rule of signs to determine the possible number of **positive** zeros of $f(x)$.

Recall that in order to find the **positive** zeros using Descartes' Rule of signs, we count the number of sign changes in our polynomial. Note that there are *two* sign changes as we count the change from a positive to a negative coefficient (from 6 to -2) and the change from negative back to positive (-2 to 5).

From this, we know that there are either 2 or $2 - 0 = 0$ positive roots.

A few of you went on to use Descartes' Rule of signs to look for negative zeros by looking at sign changes in $f(-x)$, however, this was not part of what was asked on this questtion.

7. (15 points) Find **all** zeros of the polynomial $f(x) = x^3 + 5x^2 + 9x + 5$.

First, we list all possible rational zeros of this function:

$a_0 = 5$, so possible values for p are: $p = \pm 1, \pm 5$

$a_n = 1$ so possible values for q are: $q = \pm 1$.

Then list of all possible rational zeros is given by: $\frac{p}{q} = \pm 1, \pm 5$

Checking these gives:

$$\begin{array}{r|rrrr} & 1 & 5 & 9 & 5 \\ 1 & & 1 & 6 & 15 \\ \hline & 1 & 6 & 15 & 20 \end{array}$$

$$\begin{array}{r|rrrr} & 1 & 5 & 9 & 5 \\ -1 & & -1 & -4 & -5 \\ \hline & 1 & 4 & 5 & 0 \end{array}$$

$$\begin{array}{r|rrrr} & 1 & 5 & 9 & 5 \\ 5 & & 5 & 50 & 295 \\ \hline & 1 & 10 & 59 & 300 \end{array}$$

$$\begin{array}{r|rrrr} & 1 & 5 & 9 & 5 \\ -5 & & -5 & 0 & -45 \\ \hline & 1 & 0 & 9 & -40 \end{array}$$

Acually, once we found one that worked, we really did not need to check the others.

Notice that $x = -1$ is a root. In fact, looking carefully at the result from synthetic division, we see $f(x) = (x+1)(x^2 + 4x + 5)$.

Although several of you tried some creative ways of factoring this, it does not have any additional rational roots. However, we know from the fundamental theorem of algebra, that $f(x)$ must have 3 zeros (counting multiplicities), so we use the quadratic formula to go looking for additional zeros.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{16 - 4(1)(5)}}{2(1)} = \frac{-4 \pm \sqrt{-4}}{2} = -2 \pm i.$$

Therefore, the zeros of $f(x)$ are: $x = -1$, $x = -2 - i$, and $x = -2 + i$.