Math 487 - Foundations of Geometry
Day 12 Group Assignment **Name:**________________________________

Recall: An **angle** is the union of two noncollinear rays with a common endpoint. The common endpoint is called the **vertex** of the angle, and the rays are called the **sides** of the angle.

Postulate 11:(Angle Measurement Postulate) To every angle there corresponds a real number between 0 and 180.

Postulate 12:(Angle Construction Postulate) Let $\overrightarrow{AB}$ be a ray on the edge of the half-plane H. For every r between 0 and 180, there is exactly one ray $\overrightarrow{AP}$ with P in H such that $m(\angle PAB) = r$.

Postulate 13:(Angle Addition Postulate) If D is a point in the interior of $\angle BAC$, then $m(\angle BAC) = m(\angle BAD) + m(\angle DAC)$.

Definitions:

- Two angles are **congruent** if they have the same measure, denoted $\angle ABC \cong \angle DEF$.
- The **interior of an angle** $\angle ABC$ is the intersection of set of all points on the same side of line $\overleftrightarrow{BC}$ as A and the set of all points on the same side of line $\overleftrightarrow{AB}$ as C, denoted $int(\angle ABC)$. (Note that this definition uses the Plane Separation Postulate.)
- The **interior of a triangle** $\triangle ABC$ is the intersection of the set of points on the same side of line $\overleftrightarrow{BC}$ as A, on the same side of line $\overleftrightarrow{AC}$ as B, and on the same side of line $\overleftrightarrow{AB}$ as C.
- The **bisector of an angle** $\angle ABC$ is a ray $\overrightarrow{BD}$ where D is in the interior of $\angle ABC$ and $\angle ABD \cong \angle DBC$.
- A **right angle** is an angle that measures exactly 90.
- An **acute angle** is an angle that measures between 0 and 90.
- An **obtuse angle** is an angle that measures between 90 and 180.
- Two lines are **perpendicular** if they contain a right angle.

1. Prove congruence of angles is an equivalence relation on the set of all angles.

Euclidean Angle Measure:

The measure of $\angle ABC$ in the Euclidean Plane is defined by $m(\angle ABC) = \cos^{-1}\left(\dfrac{\langle A-B, C-B\rangle}{||A-B||\,||C-B||}\right)$ where $P-Q$ is the displacement vector from point Q to point P, $\langle \star, \star \rangle$ is the dot product, and $||\star||$ is the magnitude of the vector.

2. Given the points $A(4,1)$, $B(2,3)$, and $C(0,2)$, find $m(\angle ABC)$ in the Euclidean Plane.

Note: Since points and lines are defined the same for all planes based on the Cartesian Plane, and angles are defined based on points, lines, and betweenness, angle measure in the Taxicab and Max-Distance Planes will be defined precisely as it is in the Euclidean Plane.

Poincaré **Half-Plane Angle Measure:**

The measure of $\angle ABC$ in the Poincaré Half-Plane is defined by $m(\angle ABC) = \cos^{-1}\left(\frac{\langle T_{BA}, T_{BC}\rangle}{||T_{BA}||\,||T_{BC}||}\right)$ where T_{BA} is defined as follows:

$$T_{BA} = \begin{cases} (0, y_A - y_B), & \text{if } \overleftrightarrow{AB} \text{is a Type I line.} \\ (y_B, c - x_B), & \text{if } \overleftrightarrow{AB} \text{is a Type II line } {}_c\ell_r \; x_B < x_A. \\ -(y_B, c - x_B), & \text{if } \overleftrightarrow{AB} \text{is a Type II line } {}_c\ell_r \; x_B > x_A. \end{cases}$$

3. Given the points $A(-1, 2)$, $B(3, 4)$, and $C(3, 1)$, find $m(\angle ABC)$ in the Poincaré Half-Plane.

Theorem 2.7: $m(\angle ABD) < m(\angle ABC)$ and D is on the same side of line $\overleftrightarrow{AB}$ as C if and only if $D \in int(\angle ABC)$.

4. Draw a diagram that illustrates Theorem 2.7.

5. Does every angle $\angle ABC$ have a bisector? If a bisector exists, must it be unique? Justify your answers.