Math 487 - Foundations of Geometry
Day 15 Group Assignment **Name:**________________________________

Theorem 2.11:(The Exterior Angle Theorem) Any exterior angle of a triangle is greater in measure than either of its remote interior angles.

1. Fill in reasons that justify each step in the following proof outline.

   Proof Outline: Let $\triangle ABC$ be given. Let D be a point such that $A - C - D$, and thus $\angle BCD$ is an exterior angle of $\triangle ABC$.

Statement	**Reason**
Let M be a midpoint of segment $\overline{BC}$.	
$B - M - C$ and $\overline{BM} = \overline{MC}$	
There is a point E on ray $\overrightarrow{AM}$ such that $A - M - E$ and $ME = MA$.	
$\overline{ME} \cong \overline{MA}$.	
$\angle AMB$ and $\angle EMC$ are vertical angles.	
$\angle AMB \cong \angle EMC$	
$\triangle AMB \cong \triangle EMC$	
$\angle ABC = \angle ABM \cong \angle ECM = \angle ECB$.	
$m(\angle ABC) = m(\angle ECB)$.	
E and D are on the same side of $\overleftrightarrow{BC}$.	
B, M, and E are on the same side of $\overleftrightarrow{CD}$.	
$E \in \text{int}(\angle BCD)$.	
$m(\angle BCE) + m(\angle ECD) = m(\angle BCD)$	
$m(\angle BCD) > m(\angle BCE) = m(\angle ABC)$	

The proof of the case for the other remote interior angle is similar. $\square$.

Theorem 2.12: Given a line and a point not on the line, there exists a unique line perpendicular to the given line through the given point.

Theorem 2.13: Two lines perpendicular to the same line are parallel.

2. Prove Theorem 2.13 [Hint: Use proof by contradiction and the Exterior Angle Theorem].

Theorem 2.14: There exist at least two lines that are parallel to each other.

Theorem 2.15: If there is a transversal to two distinct lines with alternate interior angles congruent, then the two lines are parallel.

Proof Outline: Assume the lines $\overleftrightarrow{AB}$, $\overleftrightarrow{DE}$, and $\overleftrightarrow{BE}$ are distinct with $A-B-C$, $D-E-F$, and $G-B-E-H$. Further, assume that A and D are on the same side of line $\overleftrightarrow{BE}$. We will prove the contrapositive statement; therefore, assume line $\overleftrightarrow{AB}$ and line $\overleftrightarrow{DE}$ are **not** parallel.

- There exists a point P at the intersection of line $\overleftrightarrow{AB}$ and line $\overleftrightarrow{DE}$.
- Without loss of generality, assume P is on the same side of line $\overleftrightarrow{BE}$ as the point A.
- Consider $\triangle PBE$
- $\angle BEF$ and $\angle CBE$ are exterior angles of $\triangle PBE$
- $m(\angle PBE) < m(\angle BEF)$ and $m(\angle PEB) < m(\angle CBE)$, i.e. $m(\angle ABE) < m(\angle BEF)$ and $m(\angle DEB) < m(\angle CBE)$.
- $\angle ABE \ncong \angle BEF$ and $\angle DEB \ncong \angle CBE$ $\square$

Definitions:

- A **Saccheri quadrilateral** is a quadrilateral $ABCD$ where $\angle BAD$ and $\angle ABC$ are right angles and $\overline{AD} \cong \overline{BC}$. The segment $\overline{AB}$ is called the **base**, and the segment $\overline{CD}$ is called the **summit**.
- A **parallelogram** is a quadrilateral in which both pairs of opposite sides are parallel.
- A **rectangle** is a quadrilateral with four right angles.

3. Draw a diagram representing a Saccheri quadrilateral.

Theorem 2.16: The diagonals of a Saccheri quadrilateral are congruent.

4. Prove Theorem 2.16.

Theorem 2.17: The summit angles of a Saccheri quadrilateral are congruent.

Proof: Let $ABCD$ be a Saccheri quadrilateral with right angles $\angle BAD$ and $\angle ABC$, and $\overline{AD} \cong \overline{BC}$. By Theorem 2.16, $\overline{AC} \cong \overline{BD}$. Since $\overline{AD} \cong \overline{BC}$, $\overline{AC} \cong \overline{BD}$, $\overline{CD} \cong \overline{DC}$, by SSS congruence [which we have not proven yet], we have $\triangle ACD \cong \triangle BDC$. Hence $\angle ADC \cong \angle BCD$. Therefore, the summit angles of a Saccheri quadrilateral are congruent. $\square$.

Theorem 2.18: The segment joining the midpoints of the base and summit of a Saccheri quadrilateral is perpendicular to both the base and summit.

Note: You should take time to read the proof of Theorem 2.18 in your textbook between now and class on Friday.