Math 487 - Foundations of Geometry
Day 1 Group Assignment **Name:**____________________________

Introduction to Axiomatic Systems

Definition: An **axiomatic system** is a finite collection of statements, called **axioms** or **postulates** which are assumed without proof. The axioms will, by necessity, contain undefined terms (also called primitive terms).

Notes:

- In order to develop a mathematical theory based on an axiomatic system, one proves new statements, called **theorems** using only the axioms, an agreed upon logical system, and previously proven theorems.
- One can make statements in a mathematical theory more concise by introducing **defined terms**
- Throughout this course, we will assume and use the following axiomatic and logical structures:
  - The Real Number System
  - Set Theory
  - Aristotelian Logic
  - The English Language
- As we progress through this course, we will begin with a few basic axioms and will continue add additional axioms. Logically speaking, as long as our axioms do not contradict each other, theorems that are true in a simpler structure with fewer axioms will remain true when additional axioms are added.

Warning: When proving geometric theorems, we will often make use of illustration or diagrams to help build intuition about what it true. However, for a proof to be valid, it must rely only on the underlying axiomatic system and basic logical principles. Diagrams can be illuminating or deceiving.

Undefined Terms: Undefined terms generally have two main types:

- **elements** are undefined terms that refer to "objects".
- **relations** are undefined terms that imply relationships between one or more objects.

A Few More Definitions:

- An Axiomatic System is **consistent** if there is no statement such that both the statement (p) and its negation ($\neg p$) are true in the axiomatic system. Otherwise, we say the system is **inconsistent**.
- An Axiomatic System is **independent** if no axiom follows as a theorem from the other axioms.
- A **model** for an axiomatic system is an assignment of meaning to the undefined terms in the system which results in the axioms all being true statements.
  - A model is **concrete** if the elements and relations are drawn from the real world.
  - A model is **abstract** if the elements and relations are taken from some other axiomatic development.
- Showing that a given axiomatic system has a model is the most common way to show that it is *consistent* (why does knowing that a model exists guarantee that it is consistent?)
  - An axiomatic system is a concrete model is said to be **absolutely consistent**.
  - An axiomatic system is an abstract model is said to be **relatively consistent**.
- An axiomatic system is said to be **complete** if every statement that can be expressed using the defined and undefined terms of the system can be proven to be either valid or invalid.

Example: Consider the following Axiomatic System.
Undefined Terms: vertex, adjacent, color
Axioms:

- **Axiom 1:** There are exactly 5 vertices.
- **Axiom 2:** If vertex "A" is adjacent to vertex "B", then vertex "B" is adjacent to vertex "A".
- **Axiom 3:** If vertex "A" is *not* adjacent to vertex "B", then there is a vertex "C" such that vertex "A" is adjacent to vertex "C" and vertex "C" is adjacent to vertex "B".
- **Axiom 4:** Each vertex is assigned exactly one of two possible colors.
- **Axiom 5:** Any pair of adjacent vertices must be assigned different colors.

1. Which undefined terms in the axiomatic system are elements? Which are relations?

2. Make a conjecture: do you think that this axiomatic system is consistent or inconsistent? Why? How could you check?

3. Find a model for this axiomatic system. Be clear about how you are assigning meaning to the undefined terms. Make sure that all five axioms are satisfied by your model.

4. Is your model concrete or abstract? How did you decide this?

5. If possible, find a second model for this axiomatic system that is different from the first one that you found. What does this tell you about this axiomatic system?

6. Do you think that the axioms in this system are independent? Why? How could you check?