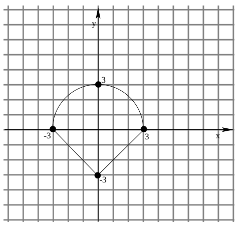
Math 487 - Foundations of Geometry
Day 20 Group Assignment **Name:**______________________________

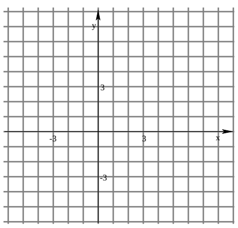
Transformational Geometry

In Chapter 2, we looked exclusively at static objects (objects that are fixed in one position). In this chapter, we will take a more dynamic approach. We will consider the idea of moving one object onto another similar object. The concept of a transformation is a function that maps a set onto another set. In some sense, motion is introduced using one-to-one and onto functions, called transformations.

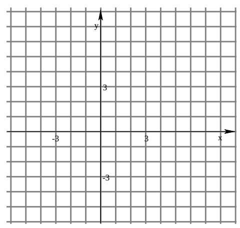
1. Draw the result after performing each indicated transformation on the figure given below.



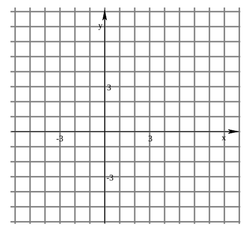
(a) Rotation 90° Counterclockwise



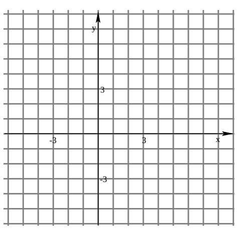
(b) Rotation by 180°



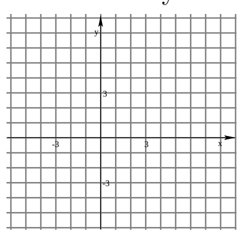
(c) Reflection across the x-axis



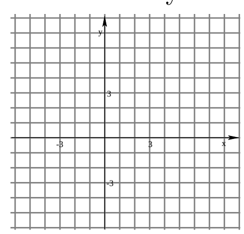
(d) Reflection across the y-axis



(e) Translation by the vector $\langle 3, 0\rangle$



(f) Translation by the vector $\langle -2, 2\rangle$



Definitions:

- A **mapping** (or function $f : A \to B$) of a set A into a set B is a rule that pairs each element of A with exactly one element of a subset of B. The set A is called the **domain**, the set B is called the **co-domain**, and the set of all elements of B (a subset of B) that are paired with an element from A is called the **range**.
- A mapping f from A to B is **onto** B if for any $b \in B$ there is at least one $a \in A$ such that $f(a) = b$.
- A mapping $f : A \to B$ is **one-to-one** if each element of the range of f is the image of exactly one element from A.
- A **transformation** is a one-to-one mapping of a set A onto a set B.
- A **transformation of a plane** is a transformation that maps points of the plane onto points in the plane.
- A nonempty set G is said to form a **group under a binary operation** $\star$, if it satisfies the following conditions:
  - If A and B are in G, then $A \star B$ is in G. (The set is **closed** under the operation, **closure**.)
  - There exists an element $I \in G$ such that for every element $A \in G$, $I \star A = A \star I = A$. (The set has an **identity**.)
  - For every element $A \in G$, there is an element $B \in G$ such that $A \star B = B \star A = I$, denoted A^{-1}. (Every element has an **inverse**.)
  - If $A, B, C \in G$, then $(A \star B) \star C = A \star (B \star C)$. (the operation $\star$ is **associative**)

Theorem 3.0: The set of transformations of a plane is a group under composition.

Proof Outline: The result follows from the following:

- The composition of two transformations of a plane is a transformation (Exercise 3.4).
- The inverse of a transformation is a transformation (Exercise 3.5).
- The identity function is a transformation and composition of functions is associative (Exercise 3.5).

2. Determine whether or not the mapping $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{x-3}{2}$ is a transformation.

3. Determine whether or not the mapping $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is a transformation.

4. Determine whether or not the mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (2x, 3y)$ is a transformation.

5. Prove the composition of two transformations of a plane is a transformation of the plane.

Note: The three parts of Exercise 3.5 (See 3.2.1 in your textbook) are presentation eligible problems for Friday in class.