Math 487 - Foundations of Geometry
Day 21 Group Assignment **Name:**____________________________

An Analytic Model of the Euclidean Plane

Our goal in this section is to find a way to define transformations of the Euclidean Plane in a "nice" matrix form. We will do so by modifying the model we used in Chapter Two. Our new model is motivated by the standard equation for a line in the Euclidean plane: $ax + by + c = 0$ where a and b are not both zero. We can rewrite this equation as the matrix equation:

$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$ where a and b are not both zero.

Since the coefficients a, b, and c define the line, the row matrix $[a, b, c]$ represents the line and the column matrix $[x, y, 1]^T$ (the superscript T indicates the transpose of the matrix) represents the points that satisfy the equation. Hence, for this model a line is defined by an ordered triplet or row matrix $[a, b, c]$ and a point is defined by an ordered triplet or column matrix, $[x, y, 1]^T$. In order to make it easier to write, we will denote a point by $(x, y, 1) = [x, y, 1]^T$.

One significant problem with the representations used in this model is that lines do not have a unique form. Notice that the two equations $2x+3y+5 = 0$ and $6x+9y+15 = 0$ describe the same line. In fact, all equations of the form $2kx+3ky+5k = 0$, where k is a nonzero real number, describe the same line as $2x + 3y + 5 = 0$.

To account for this, we must define the equivalence relation
$[a_1, a_2, a_3] \sim [b_1, b_2, b_3]$, if $b_i = ka_i$, $i = 1, 2, 3$ for some $k \neq 0$

With this in mind, we will modify the above definition of a line to be the equivalence class of ordered triples (row matrices) of the form $[a_1, a_2, a_3]$ where a_1 and a_2 are not both zero.

Since the representations of lines and points are motivated by a homogeneous matrix equation, we call the row matrix $[a_1, a_2, a_3]$, the **homogeneous coordinates of a line**, and the column matrix $(x_1, x_2, 1)$, the **homogeneous coordinates of a point**.

1. Consider the line $y = -\frac{2}{3}x + \frac{1}{6}$.

   (a) Find three different homogeneous coordinate representations of this line. At least one representation must consist of all integer entries. At least one must have 1 as one of its entries.

   (b) Find two points that are on this line and express them in homogeneous coordinate form.

   (c) Use matrix multiplication on homogeneous coordinates to show that both points are contained in the given line.

We summarize the definitions of the terms point, line, and incident for this model of the Euclidean plane as follows.

point	A column matrix denoted by $(x_1, x_2, 1)$.
line	An equivalence class of row matrices $[a_1, a_2, a_3]$ where a_1 and a_2 cannot both be zero.
incident	A point $X(x_1, x_2, 1)$ is incident with a line $\ell[\ell_1, \ell_2, \ell_3]$ iff $\ell X = 0$.

What happens with this model when three distinct points are collinear? Let $(x_1, x_2, 1), (y_1, y_2, 1)$, and $(z_1, z_2, 1)$ be three distinct collinear points. Since the points are collinear, there is a line $[a_1, a_2, a_3]$ that all three points satisfy, i.e.,

$$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

or

$$\begin{aligned} a_1x_1 + a_2x_2 + a_3 &= 0 \\ a_1y_1 + a_2y_2 + a_3 &= 0 \\ a_1z_1 + a_2z_2 + a_3 &= 0 \end{aligned}$$

or

$$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

From linear algebra, a homogeneous equation has a nontrivial solution $[a_1, a_2, a_3]$ if and only if $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ 1 & 1 & 1 \end{vmatrix} = 0$

We have proven one direction of the following theorem (completing Exercise 3.17 would complete the proof).

Proposition 3.1: Three distinct points $(x_1, x_2, 1)$, $(y_1, y_2, 1)$, and $(z_1, z_2, 1)$ are collinear iff the determinant $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ 1 & 1 & 1 \end{vmatrix} = 0$

Note: Proposition 3.1 implies a line through two distinct points $(a_1, a_2, 1)$, and $(b_1, b_2, 1)$ may be written as $\begin{vmatrix} x_1 & a_1 & b_1 \\ x_2 & a_2 & b_2 \\ 1 & 1 & 1 \end{vmatrix} = 0$

Recall: A set of lines is **concurrent** if the lines have a common point of intersection.

Proposition 3.2: Three distinct lines ℓ, m, and n are all concurrent or all parallel iff the determinant $\begin{vmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} = 0$

Note that Proposition 3.2 implies that a point on two distinct lines $[p_1, p_2, p_3]$, and $[q_1, q_2, q_3]$ may be found from the equation $\begin{vmatrix} \ell_1 & \ell_2 & \ell_3 \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix} = 0$ where ℓ represents an arbitrary unknown line.

Example: Find the point of intersection of lines $[1, 1, 1]$ and $[2, 1, -1]$.

$$\begin{vmatrix} \ell_1 & \ell_2 & \ell_3 \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = 0 \text{ iff } -2\ell_1 + 3\ell_2 - \ell_3 = 0 \text{ iff } 2\ell_1 - 3\ell_2 + \ell_3 = 0 \text{ iff } \begin{bmatrix} \ell_1 & \ell_2 & \ell_3 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}.$$

Hence the point of intersection is $(2, -3, 1)$.

Note: The third position of the point in this model must be a one. If it is not a one, then form the equivalent column matrix that has a one in the third position.

2. Select two non-parallel (and non-vertical) lines. Represent them using homogeneous coordinates. Use the matrix method demonstrated above to find their point of intersection. Then, verify this result using algebra.