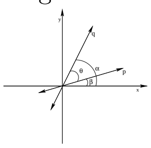
Math 487 - Foundations of Geometry
Day 22 Group Assignment **Name:**______________________________

Computing Distance in our Analytic Model: The formula for the distance between two points $P(x_1, y_1, 1)$ and $Q(x_2, y_2, 1)$ is the usual Euclidean distance formula: $d(P,Q) = \sqrt{(x_1 - x_2)^2 - (y_1 - y_2)^2}$.

Computing Angle Measure: To determine how to compute the measure of the angle between two lines $p[p_1, p_2, p_3]$ and $q[q_1, q_2, q_3]$, we consider the following.

Let $\theta = \alpha - \beta$, where α and β are the measures of the angles formed by the x-axis and the lines p and q respectively (see the diagram below).



Recall that the slopes of lines through the origin can be described using tangent: $\tan\alpha = -\dfrac{q_1}{q_2}$ and $\tan\beta = -\dfrac{p_1}{p_2}$. From this, we use the standard trigonometric identity for tangent of the difference of two angles to obtain:

$$\tan\theta = \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = \frac{-\frac{q_1}{q_2} + \frac{p_1}{p_2}}{1 + \frac{q_1}{q_2}\cdot\frac{p_1}{p_2}} = \frac{p_1q_2 - p_2q_1}{p_2q_2 + p_1q_1}$$

Therefore, the angle between two lines p and q is given by $m(\angle(p,q)) = \tan^{-1}\left(\dfrac{p_1q_2 - p_2q_1}{p_2q_2 + p_1q_1}\right)$, where $-\frac{\pi}{2} < m(\angle(p,q)) < \frac{\pi}{2}$, if $p_1q_1 + p_2q_2 \neq 0$ and $m(\angle(p,q)) = \frac{\pi}{2}$ if $p_1q_1 + p_2q_2 = 0$. Note that this method works for *any pair of lines*, not just lines through the origin.

1. Find the angle between the lines $[1, -2, -2]$ and $[3, -1, -6]$.

Affine Transformation of the Euclidean Plane

Our next goal is to investigate the form of a transformation matrix in our model. Let $A = [a_{ij}]$ be a transformation matrix on our model of the Euclidean plane and let $(x, y, 1)$ be any point in the Euclidean plane. Then we have:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11}x + a_{12}y + a_{13} \\ a_{21}x + a_{22}y + a_{23} \\ a_{31}x + a_{32}y + a_{33} \end{bmatrix}$$

2. Note that the final matrix in the equation above must represent a point in the Euclidean plane, so we must have $a_{31}x + a_{32}y + a_{33} = 1$ for every point (x, y, 1) in the Euclidean plane.

   (a) Set $(x, y, 1) = (0, 0, 1)$ in the equation above. What does this allow you to conclude about a_{33}?
   Substitute this value into the transformation matrix.

   (b) Set $(x, y, 1) = (0, 1, 1)$ in the equation above. What does this allow you to conclude about a_{32}?
   Substitute this value into the transformation matrix.

(c) Set $(x, y, 1) = (1, 0, 1)$ in the equation above. What does this allow you to conclude about a_{31}? Substitute this value into the transformation matrix.

This motivates the following definition.

Definition: An **affine transformation of the Euclidean plane**, T, is a mapping that maps each point X of the Euclidean plane to a point $T(X)$ of the Euclidean plane defined by $T(X) = AX$ where $det(A)$ is nonzero and $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \_\_ & \_\_ & \_\_ \end{bmatrix}$. where each a_{ij} is a real number.

(d) Fill in the missing row of numbers in this definition based on your work above.

Proposition 3.3: An affine transformation of the Euclidean plane is a transformation of the Euclidean plane.

3. What would we need to show in order to prove Proposition 3.3? [We may circle back and prove this later...]

Proposition 3.4: The set of affine transformations of the Euclidean plane form a group under matrix multiplication.

Proof Sketch.

4. What matrix serves as the identity element, and how do we know this matrix represents a transformation?

5. Show that the product of two transformations is a transformation.

6. How do we know that every transformation has an inverse?