Math 487 - Foundations of Geometry
Day 23 Group Assignment **Name:**______________________________________

- A transformation which preserves distance between points is an **isometry**.
- The prefix **iso** means same, equal, or identical and **metry** means distance. Therefore, the term isometry means equal distance, which is how we have defined the term.
- A property which is preserved under a transformation is said to be **invariant** under the transformation.
- *Terminology:* A transformation of a plane in a neutral geometry will be called a **transformation of a neutral plane**. A transformation of a plane in a Euclidean geometry will be called a **transformation of a Euclidean plane**. (Remember that a neutral geometry includes both Euclidean and hyperbolic geometries. See Chapter 2.)
- Two sets of points are said to be **congruent** provided there is an isometry where one set is the image of the other set. We write $\alpha \cong \beta$ if and only if there is an isometry f such that $f(\alpha) = \beta$.

1. Determine whether or not the following transformations are isometries.

   (a) $f(x, y) = (x - 2, y + 1)$.

   (b) $g(x, y) = (2x, 3y)$.

2. Describe, in words, how each of these two transformations "moves" a typical point in the plane.

Theorem 3.1: Betweenness of points is invariant under an isometry of a neutral plane.

Proof: Let f be an isometry of a neutral plane. Let A, B, and C be three distinct points such that $A - B - C$. Let $A' = f(A)$, $B' = f(B)$, and $C' = f(C)$. Using the definition of betweenness, we must have $AC = AB + BC$ and $\{A, B, C\}$ is a collinear set. Since f is an isometry, $A'C' = AC$, $A'B' = AB$, and $B'C' = BC$. Hence, $A'C' = AC = AB + BC = A'B' + B'C'$. Thus, by the Triangle Inequality, $\{A', B', C'\}$ is a collinear set . Therefore, B' is between A' and C'. $\square$.

Using Theorem 3.1 (along with and similar techniques to those used in its proof), the following can be shown to be invariant under isometry (See Corollaries 3.2 - 3.7 in your textbook for more details)

- Collinearity is invariant under an isometry of a neutral plane.
- The image of a ray under an isometry of a neutral plane is a ray.
- The image of a line segment under an isometry of a neutral plane is a congruent line segment.
- The image of a triangle under an isometry of a neutral plane is a congruent triangle.
- Angle measure is invariant under an isometry of a neutral plane.
- The image of a circle under an isometry of a neutral plane is a congruent circle.

Theorem 3.8: The set of isometries of a plane is a group under composition.

Theorem 3.9: An isometry of a Euclidean plane that fixes three non-collinear points (i.e. that maps each of these back to themselves) is the identity transformation.

Notes:

- The proof of Theorem 3.8 is presentation eligible for Wednesday in class.
- We will look at the proof of Theorem 3.9 in more detail in class on Friday.

Proposition 3.5. Collinearity is invariant under an affine transformation of the Euclidean plane.

3. Explain why (and how) Proposition 3.5 differs from the fact that collinearity is invariant under isometry.

Proof: Let A be a matrix of an affine transformation of the Euclidean plane. Assume X, Y, and Z are distinct points. Let $X' = AX$, $Y' = AY$, and $Z' = AZ$. Then:

$$\begin{bmatrix} x_1' & y_1' & z_1' \\ x_2' & y_2' & z_2' \\ 1 & 1 & 1 \end{bmatrix} = A \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ 1 & 1 & 1 \end{bmatrix}.$$

Taking the determinant of both sides of the equation, we obtain:

$$\begin{vmatrix} x_1' & y_1' & z_1' \\ x_2' & y_2' & z_2' \\ 1 & 1 & 1 \end{vmatrix} = |A| \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ 1 & 1 & 1 \end{vmatrix}.$$

Since A is the matrix of an affine transformation of the Euclidean plane, $det(A)$ is nonzero. Hence, by Proposition 3.1, the distinct points X, Y, and Z are collinear if and only if the points X', Y', and Z' are collinear . Therefore, collinearity is invariant under an affine transformation of the Euclidean plane. □

This result allows us to determine a matrix equation for determining lines.

Proposition 3.6: If A is the matrix of an affine transformation of the Euclidean plane, then the image of a line ℓ under this transformation is given by $k\ell' = \ell A^{-1}$ for some nonzero real number k where ℓ' is the image of ℓ.

Proof: Please read the proof of this result in your textbook.

4. Let $A = \begin{bmatrix} 2 & 4 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

   (a) Find the image of the line $p[1, 2, 3]$ under A [Hint: pick 2 points on p. Where they are mapped by A?].

   (b) Find the matrix A^{-1} and then find the value k such that $kp' = pA^{-1}$