Math 487 - Foundations of Geometry
Day 24 Group Assignment **Name:**________________________________

Recall: An **affine transformation of the Euclidean plane**, T, is a mapping that maps each point X of the Euclidean plane to a point $T(X)$ of the Euclidean plane defined by $T(X) = AX$ where $det(A)$ is nonzero and $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix}$. where each a_{ij} is a real number.

A transformation which preserves distance between points is an **isometry**.

Our next goal is to understand more about the matrix form of isometries in our analytic model. We investigate as follows. Let A be an affine transformation that is also an isometry. Let X and Y be two points with images $AX = X'$ and $AY = Y'$. Then we must have:

$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13} \\ a_{21}x_1 + a_{22}x_2 + a_{23} \\ 1 \end{bmatrix} = \begin{bmatrix} x'_1 \\ x'_2 \\ 1 \end{bmatrix}$. Similarly, $\begin{bmatrix} y'_1 \\ y'_2 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11}y_1 + a_{12}y_2 + a_{13} \\ a_{21}y_1 + a_{22}y_2 + a_{23} \\ 1 \end{bmatrix}$

Since A is an isometry, $d(X,Y) = d(X',Y')$, so $\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = \sqrt{(x'_1 - y'_1)^2 + (x'_2 - y'_2)^2}$
$= \sqrt{(a_{11}x_1 + a_{12}x_2 - a_{11}y_1 + a_{12}y_2)^2 + (a_{21}x_1 + a_{22}x_2 - a_{21}y_1 + a_{22}y_2)^2}$ [skipping several algebraic steps here...]
$= \sqrt{((a_{11}^2 + a_{21}^2)(x_1 - y_1))^2 + 2(a_{11}a_{12} + a_{21}a_{22})(x_1 - y_1)(x_2 - y_2) + (a_{12}^2 + a_{22}^2)(x_2 - y^2)^2}$

From this, we see that we must have:

(1) $a_{11}^2 + a_{21}^2 = 1$ (2) $a_{12}^2 + a_{22}^2 = 1$ and (3) $a_{11}a_{12} + a_{21}a_{22} = 0$

1. Suppose $a_{11} = 0$. Using equations (1), (2), and (3) above, what are the possible values for a_{21}, a_{22}, and a_{12}?

2. Suppose $a_{11} \neq 0$. Then, using (3), we have $a_{12} = -\frac{a_{21}a_{22}}{a_{11}}$.
   (a) Substitute this into (2), simplify, and apply (1). What is the relationship between the values of a_{11} and a_{22}?

   (b) If $a_{11} = a_{22}$, what can you conclude about the relationship between the values of a_{12} and a_{21}?

   (c) If $a_{11} \neq a_{22}$, what can you conclude about the relationship between the values of a_{12} and a_{21}?

Note: Since we know that $a_{11}^2 + a_{21}^2 = 1$, there is a real number θ such that $a_{11} = \cos\theta$ and $a_{12} = \sin\theta$. Note that there are no restrictions on the values of a_{13} and a_{23}. Combining this with our work above, we have the following result.

Proposition 3.7 An affine transformation of the Euclidean plane is an isometry iff its matrix representation is:

$$\begin{bmatrix} \cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{bmatrix} \text{ (direct isometry)} \qquad \begin{bmatrix} \cos\theta & \sin\theta & a \\ \sin\theta & -\cos\theta & b \\ 0 & 0 & 1 \end{bmatrix} \text{ (indirect isometry)}$$

Corollary: The determinant of a direct isometry is 1 and the determinant of an indirect isometry is -1.

3. Prove this Corollary

Proposition 3.8: The product of the matrices of two affine direct or two affine indirect isometries of the Euclidean plane is the matrix of an affine direct isometry. Further, the product of an affine direct and an affine indirect isometry of the Euclidean plane is an affine indirect isometry of the Euclidean plane.

Proposition 3.9: The set of affine direct isometries of the Euclidean plane is a group.

Proposition 3.10: The set of affine isometries of the Euclidean plane is a group.

Note: The proofs of Propositions 3.8, 3.8, and 3.10 are presentation eligible.

Proposition 3.11: For an affine direct isometry of the Euclidean plane, the measure of the angle between two lines equals the measure of the angle between the two image lines.

Take a few moments to read the proof of this proposition in your textbook.

Proposition 3.12: For an affine indirect isometry of the Euclidean plane, the measure of the angle between the two image lines has the opposite sign of the measure of the angle between the two lines.

The proof of Proposition 3.12 is similar to the argument used to prove Proposition 3.11.

4. Investigate each of the following isometries. Your goal is to provide a clear description of how each isometry "moves" points in the plane.

(a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$