Math 487 - Foundations of Geometry
Day 25 Group Assignment **Name:**____________________________________

Recall:

- An **affine transformation of the Euclidean plane**, T, is a mapping that maps each point X of the Euclidean plane to a point $T(X)$ of the Euclidean plane defined by $T(X) = AX$ where $det(A)$ is nonzero and $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix}$. where each a_{ij} is a real number.
- A transformation which preserves distance between points is an **isometry**.
- **Proposition 3.7** An affine transformation of the Euclidean plane is an isometry is and only if its matrix representation is:
$\begin{bmatrix} \cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{bmatrix}$ (direct isometry) $\begin{bmatrix} \cos\theta & \sin\theta & a \\ \sin\theta & -\cos\theta & b \\ 0 & 0 & 1 \end{bmatrix}$ (indirect isometry)

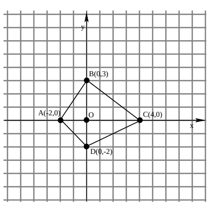
Our next goal is to understand more about the matrix form of isometries in our analytic model.
Definition: A **translation through a vector** $\overrightarrow{PQ}$ is a transformation of a plane, denoted T_{PQ}, such that if T_{PQ} maps X to X', then the vector $\overrightarrow{XX'} = \overrightarrow{PQ}$.

Definition: A **rotation about a point** C through an angle with measure θ, denoted $R_{C,\theta}$, is a transformation of a plane where C is mapped to itself and for any point X distinct from C if $R_{C,\theta}$ maps X to X', then $d(X', C) = d(X, C)$ and $m(\angle XCX') = \theta$. C is called the **center** of the rotation.

Note: By convention, the angle θ is considered to be a counter-clockwise rotation. Also, angle measure is extended to any real number value by use of the Supplement Postulate and Angle Addition Postulate. For example, a 210° angle can be formed by taking the sum of a 30°, a 150°, and another 30° angle.

Definition: A transformation f is a **symmetry** of a point set if and only if the point set is invariant under the transformation.

1. Draw the resulting image after applying each given transformation to the given geometric figure.



(a) T_{AB} (b) T_{CD} (c) $R_{O,90}$ (d) $R_{D,270}$

2. For each given isometry, draw an example of a collection of points that is invariant under that isometry.

(a) T_{AD} (b) $R_{O,180}$

Theorem 3.11: A translation of a Euclidean plane is an isometry.

Proof: Let T_{PQ} be a translation of a Euclidean plane. Let X and Y be two distinct points in the plane and $X' = T_{PQ}(X)$ and $Y' = T_{PQ}(Y)$. By the definition of a translation, $\overline{PQ}$, $\overline{XX'}$, and $\overline{YY'}$ are congruent and parallel. Two cases are possible: either X, X', Y, and Y' are not collinear or they are collinear.

Case 1: Assume X, X', Y, and Y' are not collinear. Since $\overline{XX'}$, and $\overline{YY'}$ are congruent and parallel, the quadrilateral $\square XX'Y'Y$ is a parallelogram. Hence, $XY = X'Y'$.

Case 2: Assume X, X', Y, and Y' are collinear. Suppose the points are in the order X, Y, X', Y'. Then, by betweenness of points and substitution, $XY = XX'YX' = YY'X'Y = X'Y'$. The subcases for other orders of the points are similar.

In both cases $XY = X'Y'$ for any two distinct points X and Y. Therefore, T_{PQ} is an isometry. $\square$

Theorem 3.12: A nonidentity translation has no invariant points.

Theorem 3.13: The set of translations of a plane is a group under composition.

Theorem 3.14: There exists a unique translation mapping X to Y for any two distinct points X and Y in a Euclidean plane.

Theorem 3.15: Let P and Q be two distinct points in a Euclidean plane. Line $\overleftrightarrow{PQ}$ and all lines parallel to line $\overleftrightarrow{PQ}$ are invariant under the translation T_{PQ}. No other lines are invariant.

Theorem 3.16: A rotation of a Euclidean plane is an isometry.

Note: Take a few moments to read the proofs of Theorems 3.15 and 3.16 given in your textbook.

Theorem 3.17. A nonidentity rotation has exactly one invariant point.

Theorem 3.18. The set of rotations with center C of a plane is a group under composition.

Proposition 3.13. (a) An affine translation of the Euclidean plane is a direct isometry. (b) Any affine direct isometry of the Euclidean plane with no invariant points is a translation. (c) The matrix representation of an affine translation of the Euclidean plane T_{PQ}, defined by vector $\overrightarrow{PQ} = (a, b, 0)$, is $\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$

Proposition 3.14. (a) An affine rotation of the Euclidean plane is a direct isometry. (b) Any affine direct isometry of the Euclidean plane with one invariant point is a rotation. (c) The matrix representation of an affine rotation of the Euclidean plane with center O(0, 0, 1) is $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) The matrix representation of an affine rotation of the Euclidean plane with center C not at O is $R_{C,\theta} = T_{OC} \circ R_{O,\theta} \circ T_{OC}^{-1}$.

3. Given $P(-2, 2)$, $Q(2, 2)$, $O(0, 0)$, and $R(2, 6)$

(a) Find the matrix for a translation that maps P to Q.

(b) Find the matrix for a rotation that maps P to Q.

(c) Find a matrix representation for a rotation that maps Q to R (You may express this as a product of matrices).