Math 487 - Foundations of Geometry
Day 26 Group Assignment **Name:**____________________________________

Definition: A **reflection in a line** ℓ is a transformation of a plane, denoted R_ℓ, such that if X is on ℓ, then $R_\ell(X) = X$, and if X is not on ℓ, then R_ℓ maps X to X' such that ℓ is the perpendicular bisector of $\overline{XX'}$. The line ℓ is called the **axis** of the reflection.

Definition: A **glide reflection**, denoted G_{PQ}, is the composition of the reflection with axis $\ell = \overleftrightarrow{PQ}$ and the nonidentity translation T_{PQ}, i.e. $G_{PQ} = R_\ell \circ T_{PQ}$.

Theorem 3.19: A reflection of a neutral plane is an isometry.

Proof: Let R_ℓ be a reflection of a neutral plane. Let X and Y be any two distinct points with $X' = R_\ell(X)$ and $Y' = R_\ell(Y)$. We need to show $XY = X'Y'$. One of the following must be true: (1) X and Y are on the same side of ℓ. (2) X and Y are on opposite sides of ℓ. (3) One of X and Y are on ℓ. (4) Both X and Y are on ℓ.
Case 1: Assume X and Y are on the same side of ℓ. By definition of the reflection R_ℓ, ℓ is the perpendicular bisector of segments $\overline{XX'}$ and $\overline{YY'}$ Hence, there are points P and Q on ℓ such that $\overline{XP} \cong \overline{X'P}$, $\overline{YQ} \cong \overline{Y'Q}$, $\angle YQP \cong \angle Y'QP$, and $\angle XPQ \cong \angle X'PQ$. Thus, since $\overline{PQ} \cong \overline{PQ}$, we have $\triangle YQP \cong \triangle Y'QP$ by SAS. Hence, $\overline{YP} \cong \overline{Y'P}$ and $\angle YPQ \cong \angle Y'PQ$. By angle subtraction, $\angle XPY \cong \angle X'PY'$. Hence, by SAS, $\triangle XPY \cong \triangle X'PY'$. Therefore, by the definition of congruent triangles and congruent segments, $XY = X'Y'$.

1. (a) Draw a diagram that illustrates the argument in the proof of Case 1 given above.

   (b) Draw a "relabeled" diagram to illustrate Case 2.

   (c) Draw a new diagram that illustrates Case 3.

   (d) Prove Case 4.

Theorem 3.20: The inverse of a reflection of a neutral plane is the reflection itself.

Theorem 3.21: Every point on the axis of reflection is invariant under a reflection of a neutral plane.

Theorem 3.22: All lines perpendicular to the axis of reflection are invariant under a reflection of a neutral plane.

Theorem 3.23: For any two distinct points X and Y in a neutral plane, there is exactly one reflection that maps X to Y.

Theorem 3.24: A glide reflection of a Euclidean plane is an isometry.

2. Explain why knowing that reflections and translations are isometries allows us to conclude that any glide reflection is an isometry.

Theorem 3.25: The axis of a glide reflection is the only invariant line under a glide reflection of a Euclidean plane.

Theorem 3.26: A glide reflection of a Euclidean plane has no invariant points.

Proposition 3.15: (a) An affine reflection of the Euclidean plane is an indirect isometry. (b) Any affine indirect isometry of the Euclidean plane with exactly one line with all points invariant under the isometry is a reflection. (c) The matrix representation of an affine reflection of the Euclidean plane with axis $h[0, 1, 0]$ is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(d) The matrix representation of an affine reflection of the Euclidean plane with axis ℓ distinct from h is $R_\ell = T \circ R_h \circ T^{-1}$ where T is a direct isometry that maps the line h to the line ℓ.

3. Let $P = (x, y, 1)$. Find AP and explain why the result demonstrates that A is a matrix representing R_h.

4. Prove that A is an indirect isometry.

5. Explain, in your own words, why $R_\ell = T \circ R_h \circ T^{-1}$ where T is a direct isometry that maps the line h to the line ℓ.

6. Find a matrix representation for the reflection of the Euclidean plane with axis $v[1, 0, 0]$.