Math 487 - Foundations of Geometry
Day 27 Group Assignment **Name:**______________________________

Note: This assignment is designed to help prepare you for next week's exam. It will **not** be collected and graded.

Theorem 3.27: Every isometry of a Euclidean plane is the composition of at most three reflections.

Proof: Let f be an isometry of a Euclidean plane, and let X, Y, and Z be any three noncollinear points. Further, let $X' = f(X)$, $Y' = f(Y)$, and $Z' = f(Z)$. By Corollary 3.10, to complete the proof, it is sufficient to find a composition of three reflections that maps X, Y, and Z to X', Y', and Z', respectively. If X and X' are distinct, then by Theorem 3.23 there is a unique reflection R_ℓ that maps X to X'. Let $Y_\ell = R_\ell(Y)$ and $Z_\ell = R_\ell(Z)$. Similarly, if Y_ℓ and Y' are distinct, there is a unique reflection R_m that maps Y_ℓ to Y'. Since f and R_ℓ are isometries, $X'Y' = XY = X'Y_\ell$. Hence, X' is on line m, the perpendicular bisector of $\overline{Y'Y_\ell}$ and $X' = R_m(X')$. Thus, the composition $R_m \circ R_\ell$ maps X to X', Y to Y', and Z to some point $Z_{\ell m}$. As before, if $Z_{\ell m}$ and Z' are distinct, there is a unique reflection R_n that maps $Z_{\ell m}$ to Z'. Since f, R_ℓ, and R_m are isometries, $X'Z' = XZ = X'Z_\ell = X'Z_{\ell m}$ and $Y'Z' = YZ = Y_\ell Z_\ell = Y'Z_{\ell m}$. Hence, X' and Y' are on line n, the perpendicular bisector of $\overline{Z'Z_{\ell m}}$, $X' = R_n(X')$ and $Y' = R_n(Y')$. Thus, the composition $R_n \circ R_m \circ R_\ell$ maps X to X', Y to Y', and Z to Z'. If $X = X'$, $Y_\ell = Y'$, or $Z_{\ell m} = Z'$, then we omit R_ℓ, R_m, or R_n, respectively, from the composition. By Theorem 3.20, the identity transformation is the composition of a reflection with itself. Therefore, every isometry of a Euclidean plane is the composition of no more than three reflections. □

Theorem 3.28: Every isometry of a Euclidean plane is a translation, rotation, reflection, or glide reflection.

1. Explain why Theorem 3.28 is a consequence of Theorem 3.27. [Hint: What happens when you compose two rotations with different centers?]

2. Work on proving the following proofs in your groups. Once you feel you have a good argument, any member of your group can volunteer to present a problem to the rest of the class.

   (a) **Proposition 3.8:** The product of the matrices of two affine direct or two affine indirect isometries of the Euclidean plane is the matrix of an affine direct isometry. Further, the product of an affine direct and an affine indirect isometry of the Euclidean plane is an affine indirect isometry of the Euclidean plane.

   (b) **Proposition 3.9:** The set of affine direct isometries of the Euclidean plane is a group.

   (c) **Proposition 3.10:** The set of affine isometries of the Euclidean plane is a group.

   (d) **Theorem 3.12:** A nonidentity translation has no invariant points.

   (e) **Theorem 3.13:** The set of translations of a plane is a group under composition.

   (f) **Theorem 3.14:** There exists a unique translation mapping X to Y for any two distinct points X and Y in a Euclidean plane.

   (g) **Theorem 3.18:** The set of rotations with center C of a plane is a group under composition.

   (h) **Theorem 3.20:** The inverse of a reflection of a neutral plane is the reflection itself.

   (i) **Theorem 3.26:** A glide reflection of a Euclidean plane has no invariant points.

3. (a) Find homogeneous coordinates for the line $\ell[l_1 \ l_2 \ l_3]$ containing the points $(-3, 1, 1)$ and $(1, -2, 1)$.

(b) Find the point of intersection of the lines $[4 \ -1 \ 0]$ and $[3 \ 2 \ -10]$.

(c) Find the angle between the lines $[4 \ -1 \ 0]$ and $[3 \ 2 \ -10]$.

4. (a) Find a matrix representation for the transformation T_{PQ} given:

i. $P(2, 3, 1)$, $Q(5, 3, 1)$

ii. $P(2, 3, 1)$, $Q(-1, 5, 1)$

(b) Find a matrix representation for the transformation $R_{C,\theta}$ given:

i. $C(0, 0, 1)$ and $\theta = 135°$

ii. $C(-1, 2, 1)$ and $\theta = 30°$

(c) Find a matrix representation for R_ℓ given:

i. $\ell[1 \ 0 \ 0]$

ii. $\ell[1 \ 1 \ 1]$