Math 487 - Foundations of Geometry
Day 28 Group Assignment **Name:**______________________________

Introduction to Plane Projective Geometry

Axiomatic System:

Undefined Terms: point, line, incident

Axiom 1: Any two distinct points are incident with exactly one line.

Axiom 2: Any two distinct lines are incident with at least one point.

Axiom 3: There exist at least four points, no three of which are collinear.

Axiom 4: The three diagonal points of a complete quadrangle are never collinear.

Axiom 5: (Desargues' Theorem) If two triangles are perspective from a point, then they are perspective from a line.

Axiom 6: If a projectivity on a pencil of points leaves three distinct points of the pencil invariant, it leaves every point of the pencil invariant.

Basic Notation:

- Points are denoted using upper case letters (e.g. A, B) and lines using lower case letters (e.g. a, b).
- Denote the line determined by two distinct points A and B by AB.
- Denote a point determined by two distinct lines a and b by $a \cdot b$. Note that this refers to the point of intersection of the two lines, not the set of points of intersection, $a \cap b = \{a \cdot b\}$.

1. Explain the difference between $a \cdot b$ and $a \cap b = \{a \cdot b\}$

Basic Definitions:

- A set of points is **collinear** if every point in the set is incident with the same line. Points incident with the same line are said to be **collinear**.
- Lines incident with the same point are said to be **concurrent**.
- A **complete quadrangle** is a set of four points, no three of which are collinear, and the six lines incident with each pair of these points. The four points are called **vertices** and the six lines are called **sides** of the quadrangle.
- A **diagonal point** of a complete quadrangle is a point incident with opposite sides of the quadrangle.

2. Draw a model of a complete quadrangle in a plane with points A, B, C, and D. Be sure to include all six sides

3. Extend the lines in the model of a complete quadrangle you drew above to show the three diagonal points (E, F, and G).

- A **triangle** is a set of three noncollinear points and the three lines incident with each pair of these points. The points are called **vertices**, and the lines are called **sides**.

  Note: Triangles in projective geometry different from those in Euclidean geometry. Each side of a projective triangle is a *line*, whereas each side of a Euclidean triangle is a *segment*. *Betweenness* of points is not defined in projective geometry, so projective geometry does not have segments. Despite this, we will often draw Euclidean triangles in illustrations.

- Two figures are **perspective from a point** provided the lines determined by corresponding points are concurrent. The point of concurrence is called the **center**.
- Two figures are **perspective from a line** provided the points of intersection of corresponding sides are collinear. The line containing all points of intersection is called the **axis**.

4. Draw a pair of triangles $\triangle ABC$ and $\triangle A'B'C'$ that are perspective from a point P.

5. Draw a pair of triangles $\triangle ABC$ and $\triangle A'B'C'$ that are perspective from a line ℓ.

- A set of points incident with a line is called a **pencil of points** (or *range of points*), and the line is called the **axis**.
- A set of lines incident with a point is called a **pencil of lines**, and the point is called the **center**.

6. Draw an example of a pencil of four points with axis ℓ.

7. Draw an example of a pencil of four lines with center C.

- A one-to-one mapping between two pencils of points is called a **perspectivity** if the lines incident with the corresponding points of the two pencils are concurrent. The point where the lines intersect is called the **center of the perspectivity**. A perspectivity is denoted $X \overset{O}{\bar{\wedge}} X$ where O is the center of perspectivity.
- A one-to-one mapping between two pencils of points is called a **projectivity** if the mapping is a composition of finitely many perspectivities. A projectivity is denoted $X \wedge X'$.

Note: We will look at some examples of perspectivities and projectivities in the near future.