Math 487 - Foundations of Geometry
Day 2 Group Assignment **Name:**____________________________________

Recall: An **axiomatic system** is a finite collection of statements, called **axioms** or **postulates** which are assumed without proof. The axioms will, by necessity, contain undefined terms.

Claim:

- One way to show that an axiomatic system is *consistent* is to demonstrate the existence of a model for the system.
- One way to show that an axiom is *independent* is to show that there is a model for which all of the other axioms hold but the given axiom does not hold.
- The existence of two non-isomorphic models for an axiomatic system allows us to conclude that the axiomatic system is **not** complete.

1. Explain why the existence of a model for an axiomatic system shows that the system must be consistent.

2. Use models to show that an axiom from the example in your Day 1 group work is independent of the other axioms.

Definitions:

- Two models of an axiomatic system are **isomorphic** if there is a one to one correspondence between the elements that preserves all relations.
- An axiomatic system is **categorical** if every pair of models for the system are isomorphic.
- In a geometry with two undefined terms, the **dual** of a statement in the is the statement with the two terms interchanged.

Example: Consider the following Axiomatic System.

Undefined Terms: point, line, incident

Axioms:

- **Axiom 1:** There exists at least one line
- **Axiom 2:** Every line has exactly three points incident to it.
- **Axiom 3:** Not all points are incident to the same line.
- **Axiom 4:** There is exactly one line incident with any two distinct points.
- **Axiom 5:** There is at least one point incident with any two distinct lines.

Theorem 1: Two distinct lines intersect in exactly one point.

Proof:

Let p and q be distinct lines. By Axiom 5, there is a point A that is incident to both p and q. In order to obtain a contradiction, suppose that there is a point B distinct from A that is incident to both p and q. Then p and q are both incident to the points A and B. By Axiom 4, there is exactly one line incident with any two distinct points. Hence we must have $p = q$. But we began by assuming that p and q are distinct, so this is a contradiction. Thus we must have $A = B$. $\square$.

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- **Axiom 5:** There is at least one point incident with any two distinct lines.

3. State the dual of Axiom 5 in this system. How does this dual statement compare to Axiom 4?

4. Give an argument that proves that there are at least three points in this system.

5. Give an argument that proves that there are at least four points in this system.

6. There are actually more than four points in this geometry, but there are only a finite number. Make a conjecture about how many points are in this geometry. Then, provide as much evidence as you can to support your conjecture.