Math 487 - Foundations of Geometry
Day 30 Group Assignment **Name:**________________________________

Duality in Projective Geometry

Recall: Axiomatic System for Plane Projective Geometry

Undefined Terms: point, line, incident

Axiom 1: Any two distinct points are incident with exactly one line.

Axiom 2: Any two distinct lines are incident with at least one point.

Axiom 3: There exist at least four points, no three of which are collinear.

Axiom 4: The three diagonal points of a complete quadrangle are never collinear.

Axiom 5: (Desargues' Theorem) If two triangles are perspective from a point, then they are perspective from a line.

Axiom 6: If a projectivity on a pencil of points leaves three distinct points of the pencil invariant, it leaves every point of the pencil invariant.

Our goal is to show that plane projective geometry satisfies the principle of duality. Note that once we have proven a theorem, by the principal of duality, the dual of the theorem is also valid, some one proof proves two (dual) statements.
Note: We have already proved the dual of Axiom 1 (See Theorem 4.1).

1. State and prove the dual of Axiom 2.

Dual of Axiom 3: There exist at least four lines, no three of which are concurrent.

Proof: Let A, B, C, and D be four distinct points, no three collinear; the existence of these points is guaranteed by Axiom 3. Since no three of the points are collinear, by Axiom 1, there are six distinct lines $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, $\overleftrightarrow{AD}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{BD}$, and $\overleftrightarrow{CD}$.

Consider the four lines $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{CD}$, and $\overleftrightarrow{DA}$.

2. Complete the proof by using contradiction to show that no three of these four lines are concurrent [Hint: You can focus on the case involving $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, and $\overleftrightarrow{CD}$ as the other cases are similar.]

Before examining the Dual of Axiom 4, we need to define the dual of a *complete quadrangle*, which is a *complete quadrilateral*.

Definition: A **complete quadrilateral** is a set of four lines, no three of which are concurrent, and the six points incident with each pair of these lines. The four lines are called **sides** and the six points are called **vertices** of the quadrilateral. Two vertices of a complete quadrilateral are **opposite** if the line incident to both points is not a side. A **diagonal line** of a complete quadrilateral is a line incident with opposite vertices of the quadrilateral.

3. Draw a model of a complete quadrilateral in a plane with lines a, b, c, and d and vertices E, F, G, H, I, and J.

4. Extend the model of a complete quadrilateral you drew above to show the three diagonal lines of the quadrilateral.

   Note: Quadrangles and quadrilaterals are different figures and do not have direct analogues in Euclidean geometry.

   Dual of Axiom 4: The three diagonal lines of a complete quadrilateral are never concurrent.

   Proof: Let $abcd$ be a complete quadrilateral; its existence is guaranteed by the Dual of Axiom 3. Let $E = a \cdot b$, $F = b \cdot c$, $G = c \cdot d$, $H = a \cdot d$, $I = a \cdot c$ and $J = b \cdot d$. These points exist and are unique by the Dual of Axiom 1. By Axiom 1 and the definition of diagonal lines, the diagonal lines $\overleftrightarrow{EG}$, $\overleftrightarrow{FH}$, and $\overleftrightarrow{IJ}$ exist.

   We assert that these diagonal lines are not concurrent. Suppose the diagonal lines $\overleftrightarrow{EG}$, $\overleftrightarrow{FH}$, and $\overleftrightarrow{IJ}$ are concurrent. Since $\overleftrightarrow{EG} \cdot \overleftrightarrow{FH}$ would be the point of concurrency, the points I, J, and $\overleftrightarrow{EG} \cdot \overleftrightarrow{FH}$ are collinear. Since $abcd$ is a complete quadrilateral, by definition, no three of the lines $a = \overleftrightarrow{EH}$, $b = \overleftrightarrow{EF}$, $c = \overleftrightarrow{FG}$, and $d = \overleftrightarrow{GH}$ are concurrent.

   Thus, (by a proof that is the dual of our proof of the Dual of Axiom 3) E, F, G, and H are four points, no three of which are collinear. Hence, $EFGH$ is a complete quadrangle with diagonal points $\overleftrightarrow{EF} \cdot \overleftrightarrow{GH} = b \cdot d = J$, $\overleftrightarrow{EG} \cdot \overleftrightarrow{FH}$, and $\overleftrightarrow{EH} \cdot \overleftrightarrow{FG} = a \cdot c = I$. Thus, by Axiom 4, the points I, J, and $\overleftrightarrow{EG} \cdot \overleftrightarrow{FH}$ are noncollinear, which contradicts that they are collinear. Therefore, the diagonal lines of the complete quadrilateral $abcd$ are not concurrent. □

   Dual of Axiom 5: If two triangles are perspective from a line, then they are perspective from a point.

   Proof: Assume $\triangle ABC$ and $\triangle A'B'C'$ are perspective from a line with $P = \overleftrightarrow{AB} \cdot \overleftrightarrow{A'B'}$, $Q = \overleftrightarrow{BC} \cdot \overleftrightarrow{B'C'}$ and $R = \overleftrightarrow{AC} \cdot \overleftrightarrow{A'C'}$. By the definition of perspective from a line, the points P, Q and R are collinear. Let $O = \overleftrightarrow{AA'} \cdot \overleftrightarrow{BB'}$. In order to show that $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$ and $\overleftrightarrow{CC'}$ are concurrent, we only need to show that O is on the line $\overleftrightarrow{CC'}$.

   Consider triangles $\triangle RAA'$ and $\triangle QBB'$. Since P, Q, R are collinear, P is on line $\overleftrightarrow{QR}$. Since $P = \overleftrightarrow{AB} \cdot \overleftrightarrow{A'B'}$, P is on line $\overleftrightarrow{AB}$ and line $\overleftrightarrow{A'B'}$. Hence triangles $\triangle RAA'$ and $\triangle QBB'$ are perspective from point P, by the definition of perspective from a point. Thus by Axiom 5 (Desargues' Theorem), triangles $\triangle RAA'$ and $\triangle QBB'$ are perspective from a line.

   By the definition of perspective from a line, the points $C = \overleftrightarrow{RA} \cdot \overleftrightarrow{QB}$, $C' = \overleftrightarrow{RA'} \cdot \overleftrightarrow{QB'}$ and $O = \overleftrightarrow{AA'} \cdot \overleftrightarrow{BB'}$ are collinear. Hence O is on line $\overleftrightarrow{CC'}$. Thus $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$ and $\overleftrightarrow{CC'}$ are concurrent. Therefore, $\triangle ABC$ and $\triangle A'B'C'$ are perspective from point O. □

5. Draw a diagram that illustrates the proof of the Dual of Axiom 5 given above.

   Dual of Axiom 6: If a projectivity on a pencil of lines leaves three distinct lines of the pencil invariant, it leaves every line of the pencil invariant.

   Note: The proof of the Dual of Axiom 6 is a presentation eligible problem.