Math 487 - Foundations of Geometry
Day 31 Group Assignment **Name:**______________________________________

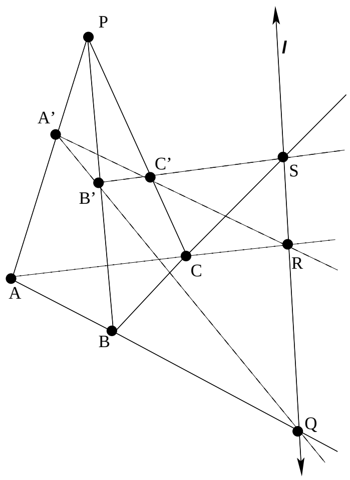
Desargues' Theorem

The French mathematician Gérard Desargues (1593 - 1662) was one of the earliest contributors to the study of synthetic projective geometry. Desargues was an engineer and architect, who had served in the French army. The importance of the theorem, that bears his name, is due to the relating of two aspects of projective geometry: perspectivity from a point and perspectivity from a line. Because of his many contributions to the field of projective geometry, the theorem was named after him even though his major work, *Brouillon projet*, was lost for nearly two centuries before another French geometer Michel Chasles (1793 - 1880) discovered a copy in 1845.

Though we are only studying plane projective geometry, a motivation for Desargues' Theorem can be seen by looking at a triangular pyramid in three dimensions. In the diagram below, we see a triangular pyramid with vertices A, B, C, and P. The triangles $\triangle ABC$ and $\triangle A'B'C'$ are perspective from the point P.

In 3-dimensional Euclidean geometry, Three non-collinear points determine a plane, and two nonparallel planes intersect in a line. Therefore, the two planes a and a' determined by $\triangle ABC$ and $\triangle A'B'C'$ intersect in a line ℓ. Since $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ are in planes a and a', respectively, the point $Q = \overleftrightarrow{AB} \cdot \overleftrightarrow{A'B'}$ must be on line ℓ.

Similarly, $R = \overleftrightarrow{AC} \cdot \overleftrightarrow{A'C'}$ and $S = \overleftrightarrow{BC} \cdot \overleftrightarrow{B'C'}$ must be on ℓ. Hence, $\triangle ABC$ and $\triangle A'B'C'$ are perspective from the line $\ell = \overleftrightarrow{QR}$.



Axiom 5: (Desargues' Theorem) If two triangles are perspective from a point, then they are perspective from a line.

Note: In plane projective geometry, Desargues' Theorem cannot be proven from the other axioms; therefore, it is taken as an axiom. The proof of the theorem requires two triangles that are not in the same plane, as illustrated in the motivating example above. That is, Desargues' Theorem can be proven from the other axioms only in a projective geometry of more than two dimensions.

Dual of Desargues' Theorem: If two triangles are perspective from a line, then they are perspective from a point.

1. Construct two triangles that are perspective from a point, then determine the line from which the triangles are perspective.

2. Construct two triangles that are perspective from a line, then determine the point from which the triangles are perspective. (You may use dynamic geometry software.)

3. Given $\triangle ABC$ and $\triangle DEF$, assume D is on $\overleftrightarrow{BC}$, E is on $\overleftrightarrow{AC}$, and F is on $\overleftrightarrow{AB}$ such that $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$, and $\overleftrightarrow{CF}$ are concurrent. Show that if $\overleftrightarrow{AB} \cdot \overleftrightarrow{DE} = P$, $\overleftrightarrow{AC} \cdot \overleftrightarrow{DF} = Q$, and $\overleftrightarrow{BC} \cdot \overleftrightarrow{EF} = R$, then P, Q, and R are collinear.

Definition: Four collinear points are said to be a **harmonic set** if there exists a complete quadrangle such that two of the points are diagonal points of the complete quadrangle and the other two points are on the opposite sides determined by the third diagonal point.

Notation: Four collinear points A, B, C, D form a harmonic set, denoted $H(AB, CD)$, if A and B are diagonal points of a complete quadrangle and C and D are on the sides determined by the third diagonal point. The point C is the **harmonic conjugate** of D with respect to A and B. Also, D is the **harmonic conjugate** of C with respect to A and B.

4. Perform the following construction in the space provided below.
   - Draw a line ℓ and add distinct points A, B, and C all on ℓ.
   - Add a point E not on ℓ
   - Add a point F on $\overleftrightarrow{AE}$ distinct from A and E.
   - Add $G = \overleftrightarrow{CE} \cdot \overleftrightarrow{BF}$.
   - Add $H = \overleftrightarrow{AG} \cdot \overleftrightarrow{BE}$.
   - Add $D = \overleftrightarrow{AB} \cdot \overleftrightarrow{FH}$.
   - Note that E, F, G, and H determine a complete quadrangle, and that $H(AB, CD)$ is a harmonic set.