Math 487 - Foundations of Geometry
Day 32 Group Assignment **Name:**______________________________

Harmonic Sets

Recall: Four collinear points are said to be a **harmonic set** if there exists a complete quadrangle such that two of the points are diagonal points of the complete quadrangle and the other two points are on the opposite sides determined by the third diagonal point.

Notation: Four collinear points A, B, C, D form a harmonic set, denoted $H(AB, CD)$, if A and B are diagonal points of a complete quadrangle and C and D are on the sides determined by the third diagonal point. The point C is the **harmonic conjugate** of D with respect to A and B. Also, D is the **harmonic conjugate** of C with respect to A and B.

1. Last time, you were asked to carry out the construction given below. Now, you should give a brief justification for each step in this construction (based on axioms, definitions, and previously proven theorems).

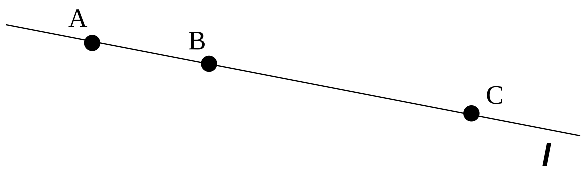
   - Draw a line ℓ and add distinct points A, B, and C all on ℓ.
   - Add a point E not on ℓ
   - Add a point F on $\overleftrightarrow{AE}$ distinct from A and E.
   - Add $G = \overleftrightarrow{CE} \cdot \overleftrightarrow{BF}$.
   - Add $H = \overleftrightarrow{AG} \cdot \overleftrightarrow{BE}$.
   - Add $D = \overleftrightarrow{AB} \cdot \overleftrightarrow{FH}$.
   - Note that E, F, G, and H determine a complete quadrangle, and that $H(AB, CD)$ is a harmonic set.

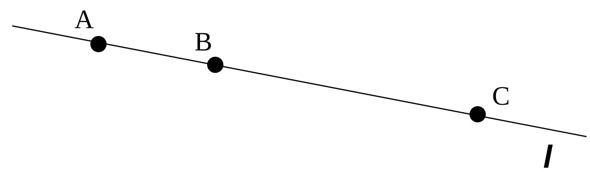
   Notes: The construction you carried out last time together with the justifications you provided above together prove the following theorems.

   - **Theorem 4.5:** There exists a harmonic set of points.
   - **Theorem 4.6:** If A, B, and C are three distinct collinear points, then a harmonic conjugate of C with respect to A and B exists.
   - There is one remaining technical detail that we need to resolve: How can we be sure that D is distinct from our original points A, B, and C? Proving that D is distinct from A, B, and C is a presentation eligible problem.

2. Note that in our notation for harmonic sets, the two points that are given first are diagonal points of the constructed complete quadrangle and the second two are the conjugate points. **Claim:** $H(AB, CD)$, $H(BA, CD)$, $H(AB, DC)$, and $H(BA, DC)$ all represent the same harmonic set of points. Explain, in your own words, why these four expressions are all equivalent.

3. Given three collinear points A, B, C, as illustrated below, use our constructive procedure to to add E ,F, G, H, and D to the first diagram. Then, make different choices for E and F and complete the construction a second time using the diagram on the right. Is the resulting point D the same or different? Compare your results with others in your group.





Theorem 4.7: If A, B, and C are three distinct collinear points, then the harmonic conjugate of C with respect to A and B is unique.

Proof: Let A, B, and C be three distinct collinear points. By Theorem 4.6, D, a harmonic conjugate of C with respect to A and B exists. Let $EFGH$ be a complete quadrangle used to construct D with $A = EF \cdot GH$, $B = EH \cdot FG$, $C = EG \cdot AB$, and $D = FH \cdot AB$. Let D' a harmonic conjugate of C with respect to A and B constructed using a different complete quadrangle $E'F'G'H'$ with $A = E'F' \cdot G'H'$, $B = E'H' \cdot F'G'$, $C = E'G' \cdot AB$, and $D' = F'H' \cdot AB$. We claim that $D = D'$.

By the definition of two triangles perspective from a line, $\triangle EFG$ and $\triangle E'F'G'$ are perspective from line $\overleftrightarrow{AB}$. Thus, by the dual of Desargues' Theorem, $\triangle EFG$ and $\triangle E'F'G'$ are perspective from a point. Hence,the lines $\overleftrightarrow{EE'}$, $\overleftrightarrow{FF'}$, and $\overleftrightarrow{GG'}$ are concurrent. Similarly, for $\triangle EGH$ and $\triangle E'G'H'$, the lines $\overleftrightarrow{EE'}$, $\overleftrightarrow{GG'}$, and $\overleftrightarrow{HH'}$ are concurrent.

Since the lines $\overleftrightarrow{EE'}$, $\overleftrightarrow{FF'}$, $\overleftrightarrow{GG'}$, and $\overleftrightarrow{HH'}$ are all concurrent, $\triangle FGH$ and $\triangle F'G'H'$ are perspective from a point.

Hence, by Desargues' Theorem, $\triangle FGH$ and $\triangle F'G'H'$ are perspective from a line. Therefore, $GH \cdot G'H' = A$, $FG \cdot F'G' = B$, and $FH \cdot F'H'$ are collinear. That is, $FH \cdot F'H'$ is on line $\overleftrightarrow{AB}$. Hence $D = FH \cdot AB = F'H' \cdot AB = D'$.

Therefore, the harmonic conjugate of C with respect to A and B is unique. $\square$.

4. Draw a diagram that illustrates the proof of Theorem 4.7 given above.