Math 487 - Foundations of Geometry
Day 33 Group Assignment **Name:**____

Recall: Four collinear points A, B, C, D form a harmonic set, denoted $H(AB, CD)$, if A and B are diagonal points of a complete quadrangle and C and D are on the sides determined by the third diagonal point. The point C is the **harmonic conjugate** of D with respect to A and B. Also, D is the **harmonic conjugate** of C with respect to A and B.

Theorem 4.7: If A, B, and C are three distinct collinear points, then the harmonic conjugate of C with respect to A and B is unique.

We will now consider one last important question about harmonic sets. Do $H(AB, CD)$, and $H(CD, AB)$ both exist? If so, are they related to each other?

Since previous theorems show that we can always find a unique fourth point given three collinear points, we know that both harmonic sets exist and we would probably guess that the two harmonic sets are the same. That is, if we began with C, D, and A, we would expect to obtain the point B using our construction. If so, then B is the harmonic conjugate of A with respect to C and D. In fact, this does turn out to be the case.

Theorem 4.8: $H(AB, CD)$ if and only if $H(CD, AB)$.

Proof: We will only show one direction of this biconditional statement, since the proof of the converse is similar. Since $H(AB, CD)$, there is a complete quadrangle $EFGH$ such that $A = EF \cdot GH$, $B = EH \cdot FG$, $C = EG \cdot AB$, and $D = FH \cdot AB$. We need a complete quadrangle which has C and D as diagonal points and where A and B are determined by the remaining pair of opposite sides.

Let $P = CF \cdot DG$ and $Q = FH \cdot EG$. Note that each set $\{F, P, C\}$, $\{F, G, B\}$, $\{F, Q, H, D\}$, $\{E, Q, G, C\}$, and $\{P, G, D\}$ are collinear sets. Furthermore, no three of F, G, P, Q are collinear. Consider the complete quadrangle $FGPQ$. Notice that $PF \cdot GQ = CF \cdot EG = C$, $FQ \cdot PG = FH \cdot DG = D$, and $FG \cdot DC = B$. We must still show that $PQ \cdot DC = A$.

Note that $\triangle EHQ$ and $\triangle FGP$ are perspective from a line, since $B = EH \cdot FG$, $C = EQ \cdot FP$, and $D = HQ \cdot GP$ are collinear. Hence by the dual of Desargues' Theorem, they are also perspective from a point. Thus, since $EF \cdot HG = A$, PQ contains A, which implies that $PQ \cdot DC = A$.

Therefore, if $H(AB, CD)$, then $H(CD, AB)$. $\square$

1. Draw a diagram that illustrates the proof of Theorem 4.8 given above.

2. Explain, in your own words, why the proof of the converse is similar to the proof given above.

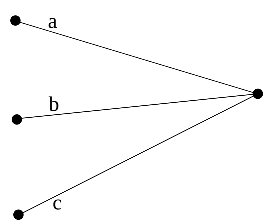
Corollary 4.9: The following are equivalent: $H(AB, CD)$, $H(AB, DC)$, $H(BA, CD)$, $H(BA, DC)$, $H(CD, AB)$, $H(CD, BA)$, $H(DC, AB)$, and $H(DC, BA)$.

3. Prove Corollary 4.9

4. Consider the dual of a harmonic set of points.

   (a) Write the definition for the dual of a harmonic set of points.

   (b) Construct the harmonic conjugate of c with respect to a and b [Hint: use the dual of the construction for harmonic sets of points].



   (c) Write the duals for Theorems 4.5, 4.6, 4.7, and 4.8.

   (d) Explain how we know that all four of these dual theorems are true.