Math 487 - Foundations of Geometry
Day 34 Group Assignment **Name:**________________________

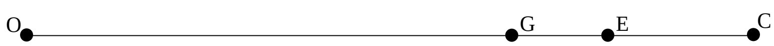
Harmonic Sets and Music

The **major diatonic scale** consists of notes with the frequency ratios $1, \frac{9}{8}, \frac{5}{4}, \frac{4}{3}, \frac{3}{2}, \frac{5}{3}, \frac{15}{8}, 2$ relative to a key note (the tonic). Though there are many different definitions and formulations of which chords are harmonic, most often, chords having the frequency ratios $1:2:3$, $2:3:4$, $3:4:5$, and $4:5:6$ are called **harmonic**.

Consider the major triad with frequency ratio $4:5:6$, which is equivalent to the ratio $1:\frac{5}{4}:\frac{3}{2}$. With a string tuned to C, the diatonic frequency ratios give the notes $1(C)$, $\frac{9}{8}(D)$, $\frac{5}{4}(E)$, $\frac{4}{3}(F)$, $\frac{3}{2}(G)$, $\frac{5}{3}(A)$, $\frac{15}{8}(B)$, $2(C)$. Note that the second 'C' (one octave up from the original tonic C) has a frequency exactly twice that of the tonic.

Hence, the ratio $4:5:6(1:\frac{5}{4}:\frac{3}{2})$ gives the notes C, E, and G. Since the period is the reciprocal of the frequency, the ratio of the lengths of the string to the corresponding notes would be $1:\frac{4}{5}:\frac{2}{3}$ for C, E, and G. We consider a string tuned to C with E occurring at $\frac{4}{5}$ and G at $\frac{2}{3}$ of the length of the string, respectively.

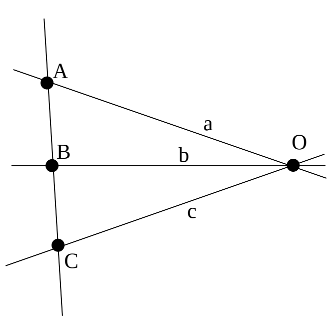
1. Use the following to create a diagram that illustrates that the points O, G, E, C form a harmonic set $H(OE, CG)$; that is, that G is the harmonic conjugate of C with respect to O and E.



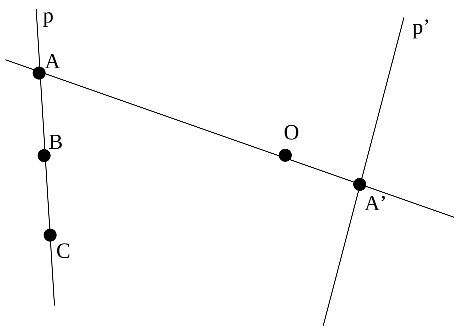
Definitions:

- A one-to-one mapping between a pencil of points and a pencil of lines is called an **elementary correspondence** if each point of the pencil of points is incident with the corresponding line of the pencil of lines. The elementary correspondence is denoted $X \barwedge x$ or $x \barwedge X$. [An elementary correspondence is also called a **perspectivity** between a pencil of points and a pencil of lines.]
- A one-to-one mapping between two pencils of points is called a **perspectivity** if the lines incident with the corresponding points of the two pencils are concurrent. The point where the lines intersect is called the **center of the perspectivity**. The perspectivity is denoted $X \overset{O}{\barwedge} X'$ where O is the center of perspectivity.
- A one-to-one mapping between two pencils of lines is called a **perspectivity** if the points of intersection of the corresponding lines of the two pencils are collinear. The line containing the points of intersection is called the **axis of the perspectivity**. The perspectivity is denoted $x \overset{o}{\barwedge} x'$ where o is the axis of perspectivity.
- A one-to-one mapping between two pencils of points is called a **projectivity** if the mapping is a composition of finitely many elementary correspondences or perspectivities. A projectivity is denoted $X \wedge X'$ or $x \wedge x'$ or $x \wedge X$ or $X \wedge x$.
- When a projectivity exists between two pencils, the pencils are said to be **projectively related**. Also, note that elementary correspondences and perspectivities themselves are projectivities.

The following diagram illustrates an elementary correspondence $ABC \barwedge abc$ between the pencil of points A, B, C, and the pencil of lines a, b, c.

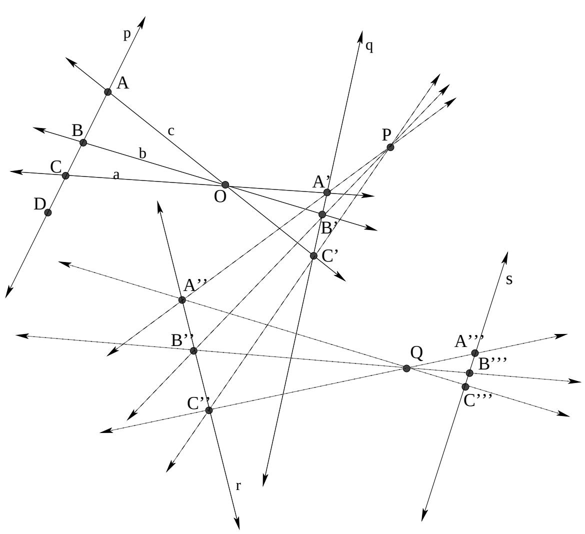


2. Complete the diagram given below to illustrate the perspectivity $ABC \overset{O}{\barwedge} A'B'C'$



3. In the space provided, illustrate a perspectivity between a pencil of lines a, b, c with point of concurrence P and a second pencil of lines a', b' , c' with point of concurrence P'.

4. Find the image of the point D in the projectivity $ABC \wedge A'''B'''C'''$ illustrated below.



5. List each of the individual perspectivities between pencils of points within the projectivity $ABC \wedge A'''B'''C'''$ shown above, using appropriate notation.