Math 487 - Foundations of Geometry
Day 35 Group Assignment **Name:**__________________________________

The Fundamental Theorem of Projective Geometry

Claim: There exists a projectivity between two pencils of points.

Assume A, B, and C are elements of a pencil of points with axis p and that A', B', and C' are elements of a pencil of points with axis p'. Also assume that all of the points are distinct and that the axes p and p' are distinct. We desire to find a projectivity $ABC \wedge A'B'C'$. Since a projectivity is a composition of perspectivities, we will accomplish this by constructing two perspectivities that together map ABC to $A'B'C'$.

To construct the first perspectivity, we define the center of a perspectivity that will map A to A' and C to itself. The image of B will also be found. Let P be a point on AA' that is distinct from A and A'. Let $B_1 = BP \cdot A'C$. Thus $ABC \overset{P}{\barwedge} A'B_1C$.

1. Explain why we know that the point P exists.

2. In the space provided below, create a diagram that illustrates two pencils of points ABC and ABC along with the perspectivity $ABC \overset{P}{\barwedge} A'B_1C$.

For the second perspectivity, we define the center of a perspectivity that maps A' to itself, B_1 to B', and C to C'. Let $Q = B_1B' \cdot CC'$. Then $A'B_1C \overset{Q}{\barwedge} A'B'C'$. Since $ABC \overset{P}{\barwedge} A'B_1C$ and $A'B_1C \overset{Q}{\barwedge} A'B'C'$, we have $ABC \wedge A'B'C'$.

3. Add an illustration of the second perspectivity $A'B_1C \overset{Q}{\barwedge} A'B'C'$ to your diagram above.

This construction proves the following theorem.

Theorem 4.10: If A, B, C and A', B', C' are distinct elements in pencils of points with distinct axes p and p', respectively, then there exists a projectivity such that $ABC \wedge A'B'C'$.

Furthermore, we can extend this constructive procedure to determine a corresponding point D' on axis p' by following the two constructed perspectivities when any distinct fourth point D on axis p is given. That is, let D be an element of axis p. We first find the point D_1 on the pencil of points with $A'B_1C$ by mapping D through center P; that is, we define $D_1 = DP \cdot A'C$. Next, we map D_1 to D' by mapping D_1 through the center Q to p'; that is, we define $D' = D_1Q \cdot p'$. Then $ABCD \overset{P}{\barwedge} A'B_1CD_1$ and $A'B_1CD_1 \overset{Q}{\barwedge} A'B'C'D'$. Hence we have $ABCD \wedge A'B'C'D'$.

4. Add a point D to your diagram above, and then find the corresponding points D_1 and D' given by applying this construction.

Recall: Axiom 6 If a projectivity on a pencil of points leaves three distinct points of the pencil invariant, it leaves every point of the pencil invariant.

Notice that Axiom 6 implies that a projectivity on a pencil that leaves three elements of the pencil invariant is the identity mapping. What implications does this axiom have for a projectivity on a pencil of points where no group of three points are mapped to themselves? Can this axiom extend the above theorem for constructing a projectivity between two pencils of points to more than three points? All of these questions are answered by the Fundamental Theorem of Projective Geometry, which proves that only three pairs of points are needed to determine a unique projectivity between two pencils of points.

Theorem 4.11: (Fundamental Theorem of Projective Geometry) A projectivity between two pencils of points is uniquely determined by three pairs of corresponding points.

In other words, if A, B, C, and D are in a pencil of points with axis p and A', B', and C' are in a pencil of points with axis p', then there exists a unique point D' on p' such that $ABCD \wedge A'B'C'D'$.

Proof: Assume A, B, C, and D are in a pencil of points with axis p and that A', B', and C' are in a pencil of points with axis p'. We have shown that there exists a point D' on p' such that $ABCD \wedge A'B'C'D'$. Suppose there is a projectivity and a point D'' such that $ABCD \wedge A'B'C'D''$. Since $A'B'C'D' \wedge ABCD$ and $ABCD \wedge A'B'C'D''$, we have $A'B'C'D' \wedge A'B'C'D''$. Therefore, by Axiom 6, $D' = D''$. $\square$.

Note: The principle of duality extends the fundamental theorem to pencils of lines. Similar arguments can be used to extend the theorem to cases where one set is a pencil of lines and the other is a pencil of points.

Corollary 4.12: A projectivity between two distinct pencils of points with a common element that corresponds to itself is a perspectivity.

Note: The proof of Corollary 4.12 is a presentation eligible problem.