Math 487 - Foundations of Geometry
Day 36 Group Assignment **Name:**________________________________

Harmonic Sets and Projectivity

Note: The Fundamental Theorem of Projective Geometry states that three pairs of corresponding points completely determine a projectivity between two pencils of points. Our next goal is to understand what happens to harmonic sets under projectivity. Since harmonic sets consist of four points (rather that three), several questions arise: *Is the projection of a harmonic set, also a harmonic set? That is, is a harmonic relationship invariant under a projectivity? Does a projectivity exist between any two harmonic sets?*

We begin by investigating whether or not an elementary correspondence preserves a harmonic set.

1. Let A, B, and C be three distinct points in a pencil of points. In the space provided below, use the standard construction to find D, the harmonic conjugate of C with respect to A and B. Be sure to indicate the underlying complete quadrangle $EFGH$ from your construction.

2. Given $H(AB, CD)$, the harmonic set of points you constructed above, define (and label) the lines $a = \overleftrightarrow{AE}$, $b = \overleftrightarrow{BE}$, $c = \overleftrightarrow{CE}$, $d = \overleftrightarrow{DE}$ in your diagram above. Note that a, b, c, d is a pencil of lines with center E where E is a point not on $\overleftrightarrow{AB}$.

3. Describe the relationship between the pencil of points A, B, C, D and the pencil of lines a, b, c, d.

4. Show that $\overleftrightarrow{FG}$, $\overleftrightarrow{FH}$, $\overleftrightarrow{AH}$, $\overleftrightarrow{AB}$ is a complete quadrilateral.

5. Explain why this underlying quadrilateral shows that $H(ab, cd)$ is a harmonic set of lines.

Notes:

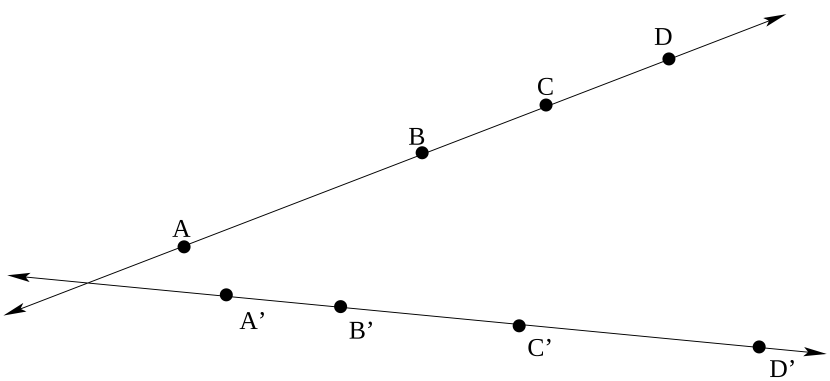
- Since the point E in our general construction is an arbitrary point not on $\overleftrightarrow{AB}$, our work above demonstrates that an elementary correspondence maps a harmonic set of points to a harmonic set of lines.
- By the principle of duality, the converse is also true. That is, an elementary correspondence maps a harmonic set of lines to a harmonic set of points.
- Since a projectivity is a finite composition of elementary correspondences, any projectivity maps a harmonic set to another harmonic set. Therefore, we have proven that a harmonic relationship is invariant under a projectivity as stated in the following theorem [we would need to rearrange our argument and quantify in the correct order to produce a correct formal proof of Theorem 4.13].

Theorem 4.13: If $H(AB, CD)$ and $ABCD \wedge A'B'C'D'$, then $H(A'B', C'D')$.

Theorem 4.14: There exists a projectivity between any two harmonic sets.

6. Prove Theorem 4.14 [Hint: Theorem 4.10 might be helpful]

7. Consider the diagram below.



   (a) Add the lines AB', BA', AC', CA', AD', and DA', BC', CB', BD', DB', CD' and DC'.

   (b) Consider the points $AB' \cdot BA'$, $AC' \cdot CA'$, $AD' \cdot DA'$, $BC' \cdot CB'$, $BD' \cdot DB'$, and $CD' \cdot DC'$. How do these points seems to be related to one another?

   (c) Add another point E to the pencil of points A, B, C, D. Based on your observations above, find the point E' that corresponds to E in the pencil of points A', B', C', D'.