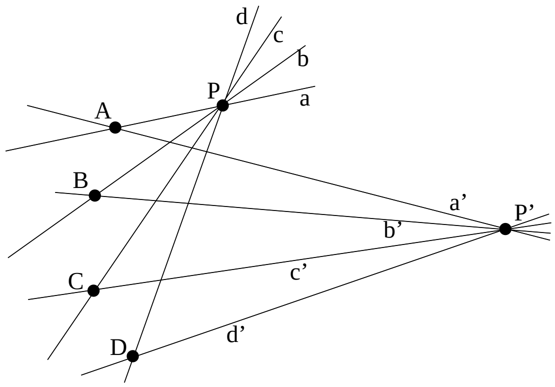
Math 487 - Foundations of Geometry
Day 37 Group Assignment **Name:**________________________________

Conics in the Projective Plane

1. Consider the two pencils of lines a, b, c, d, and a', b', c', and d' with centers P and P' respectively shown in the following diagram.



   (a) Are these two pencils perspectively related? Justify your answer.

   (b) Add additional corresponding lines e, f, g, h, and e', f', g', h' to the pencils of lines in the diagram above. Label the points of intersection of corresponding lines as E, F, G, and H.

   (c) How are the points A, B, C, D, E, F, G, and H related to each other? Describe their relationship as clearly and accurately as you can.

2. In the space provided below, draw two pencils of points A, B, C, D, E, F, and A', B', C', D', E', F' with distinct, non-parallel axes. Add a line connecting each pair of corresponding points in the two pencils (i.e. draw in AA', BB' etc.). Continue adding pairs of corresponding points and a line connecting them to your diagram until a pattern emerges. Describe the pattern that emerges in your diagram.

Definition: A **point conic** is the set of points of intersection of corresponding lines of two projectively, but not perspectively, related pencils of lines with distinct centers.

Definition: A **line conic** is the set of lines that join corresponding points of two projectively, but not perspectively, related pencils of points with distinct axes.

Theorem 4.16: The centers of the pencils of lines defining a point conic are points of the point conic.

Proof: Let P and P' be the centers of two pencils of lines defining a point conic. Let $p = PP'$. Then p is a line in the pencil with center P. Since the two pencils of lines are projectively related, there is a line p' corresponding to p in the pencil of lines with center P'. Since the pencils of lines are not perspectively related, p and p' are distinct (by the Dual of Corollary 4.12). Hence, by definition of point conic, $P' = p \cdot p'$ is a point of the point conic.

The argument is the similar for P. Let $p' = PP'$ be a line in the pencil with center P'. Then there is a distinct line p corresponding to p' in the pencil of lines with center P. Hence, $P = p \cdot p'$ is a point of the point conic. Therefore, P and P' are both points in the point conic. $\square$.

Note: Applying the dual of the Fundamental Theorem, we know that a projectivity between two pencils of lines is uniquely determined by three pairs of corresponding lines. From this, the definition of a point conic implies that given any three pairs of corresponding lines, a unique projectivity is determined. Furthermore, three points of a point conic are also determined. Combining this observation with Theorem 4.16, we obtain five distinct points in a point conic.

A natural question arises: Do any five points, where no three are collinear, determine a point conic? We investigate the question by letting A, B, C, D, E be five distinct points, no three of which are collinear.

The lines AD, BD, CD and the lines AE, BE, CE are two pencils of lines with centers D and E, respectively. By the dual of the Fundamental Theorem, there is a unique projectivity between these two pencils of lines. Since A, B, and C are intersections of corresponding lines and are noncollinear, the projectivity is not a perspectivity. Hence, by the definition of a point conic and Theorem 4.16, A, B, C, D, and E are points of a point conic. This proves the following theorem.

Theorem 4.17: Any five distinct points, no three collinear, determine a point conic where two of the points are the centers of the respective pencils of lines.

Note that any two points may be chosen as the centers of the respective pencils. Do different choices for the centers give different point conics? That is, do any five points, no three collinear, determine a unique point conic?

3. For each of the sets of five points given in the diagram below, choose two points as centers and form corresponding point conics (make "different" choices). How do the two point conics compare to one another?