Math 487 - Foundations of Geometry
Day 4 Group Assignment **Name:**____________________________

Recall: The Axiomatic System for a Finite Projective Plane:

Undefined Terms: point, line, incident
Defined Term: Points incident to the same line are *collinear*.
Axioms:

- **Axiom P1:** For any two distinct points, there is exactly one line incident with both points.
- **Axiom P2:** For any two distinct lines, there is at least one point incident with both lines.
- **Axiom P3:** Every line has at least three points incident with it.
- **Axiom P4:** There exist at least four distinct points of which no three are collinear.

Definition: A **projective plane of order n** is a geometry that satisfies the above axioms for a finite projective plane and has at least one line with exactly $n+1$ ($n > 1$) distinct points incident with it.

Theorem P2: In a projective plane of order n, there exists at least one point with exactly $n+1$ distinct lines incident with it.

Proof:

By the definition of a projective plane of order n, there exists a line ℓ with exactly $n+1$ points incident to it, call these points $P_1,\ P_2,\ \cdots, P_{n+1}$. By Axiom $P4$, there is a point Q not incident with ℓ. By Axiom $P1$, there must be lines $QP_1, QP_2, \cdots, QP_{n+1}$. We need to show these lines are distinct and that there are no other lines through Q.

1. Use proof by contradiction and axiom P1 to prove that if $i \neq j$, then $QP_i \neq QP_j$.

   To complete the proof, it remains to show that these $n+1$ lines are the only lines incident to Q. To see this, suppose that m is a line that is incident to Q. By axiom $P2$, m and ℓ must be incident at some point R. Since R is incident to ℓ, $R = P_i$ for some i. Hence, by axiom $P1$, $m = P_i$. $\square$.

2. State and prove the dual of axiom $P1$.

   Theorem P3: In a projective plane of order n, every point is incident with exactly $n+1$ lines.

   Proof:

   Let P be a point in a projective plane of order n. By the definition of a projective plane of order n, there exists a line ℓ with exactly $n+1$ points incident to it, call them $P_1, P_2, ..., P_{n+1}$. We have two possible cases: either P is incident to ℓ or P is not incident to ℓ.

   Case 1: Suppose P is *not* incident to ℓ. The proof of this case follows immediately from the proof of Theorem $P2$, taking $Q = P$. Hence, in this case, P is incident with exactly $n+1$ lines.

Case 2: Suppose P is incident to ℓ. Then $P = P_i$ for some $i \in \{1, ..., n+1\}$.

Note: Our strategy will be to show that there is a line m satisfying: P is not incident with m and m has $n+1$ distinct points, which will allow us to apply Case 1.

By Axiom $P4$, these must be distinct points Q and R both not incident with ℓ. By Axiom $P1$ and Axiom $P3$, the lines RP_1, RP_2, and RP_3 all exist. Furthermore, since R is not on ℓ, by Axiom $P1$, P is not incident to at least two of the three lines RP_1, RP_2, and RP_3. By axiom $P1$, Q is not incident to at least two of the three lines RP_1, RP_2, and RP_3. Thus at least one of the lines RP_1, RP_2, and RP_3 has neither P nor Q incident with it.

Since Q is not on ℓ, using Case 1, Q is incident to exactly $n+1$ lines: $m_1, m_2, \cdots, m_{n+1}$. By the Dual of Axiom $P1$, each line m_j intersects m at exactly one point S_j for $j = 1, 2, \cdots, n+1$. These $n+1$ points must be distinct. Otherwise, if $S_j = S_k$ for some $j \neq k$, then $m_j = QS_j = QS_k = m_k$ for $j \neq k$, which contradicts the fact that $m_1, m_2, \cdots, m_{n+1}$ are all distinct. Moreover, the $n+1$ points $S_1, S_2, \cdots, S_{n+1}$ are the only points on m. Otherwise, there would be another point T on m distinct from the $n+1$ points $S_1, S_2, ..., S_{n+1}$, but then an $(n+2)nd$ line $QT \neq m_j$, $j = 1, 2, ..., n+1$, would intersect the line m, which contradicts the fact Q is incident to exactly $n+1$ lines. Hence, there are exactly $n+1$ points on m. Therefore, P is not on m and m contains exactly $n+1$ points. Thus, we may apply Case 1 to conclude that P is incident with exactly $n+1$ lines.

By Cases 1 and 2, every point in a projective plane of order n is incident with exactly n + 1 lines. □

3. Create diagrams that help illustrate the constructive arguments in Case 2 of the proof above.

4. Prove the dual of axiom $P4$ (thus completing the proof that this axiomatic system satisfies the principle of duality).

Theorem P4: In a projective plane of order n, every line is incident with exactly $n+1$ points.

Proof: This result follows immediately from Theorem $P3$ and the fact that this axiomatic system satisfies the principle of duality.

Theorem P5: In a projective plane of order n, there exist exactly n^2+n+1 points and n^2+n+1 lines.

Proof:

Let P be a point in a projective plane of order n. The existence of P is guaranteed by Axiom P4, as is the existence of other points distinct from P. By Axiom $P1$, every point distinct from P must be on exactly one line with P. By Theorem $P3$, there are exactly $n+1$ lines incident with P. By Theorem $P4$, each of these lines is incident with exactly n points distinct from P (and, of course, distinct from each other). Therefore, there are $n(n+1)+1 = n^2+n+1$ points in the geometry.

By the principle of duality, there are also n^2+n+1 lines in the geometry. □.