Math 487 - Foundations of Geometry
Day 5 Group Assignment **Name:**______________________________

Background:
Euclid (circa 300 B.C.) was a Greek mathematician for whom Euclidean geometry was named. His most famous work is the *Elements*, a vast composition which is a consolidation of the then known concepts of mathematics and geometry, including definitions, axioms, and the proofs of over 400 theorems. The *Elements* is referred to by many as one of the most influential mathematics books of all time.

In the Elements, Euclid noted postulates and axioms that he considered common sense ideas that were assumed to be true and unquestionable. The book starts by listing 23 definitions. Euclid then listed his five postulates that he used as a base to prove the theorems. Throughout the Elements, postulates referred to assumptions specific to the study of geometry and axioms were assumptions used throughout the study of mathematics.

Euclid's Five Postulates:

- Postulate 1: To draw a straight line from any point to any point.
- Postulate 2: To produce a finite straight line continuously in a straight line.
- Postulate 3: To describe a circle with any center and radius.
- Postulate 4: That all right angles equal one another.
- Postulate 5: That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

1. Make sense of Euclid's Five Postulates by rewriting them in your own words.

Alternatives to Euclid's Fifth Postulate:

Many (including perhaps Euclid himself) found the original phrasing of the postulate to be cumbersome. It was also long suspected that Euclid's Fifth Postulate might not be independent and many mathematicians attempted to prove that is was a consequence of the other four postulates.

Proclus Diadochus (412485) wrote a commentary on the Elements. His attempt to prove the fifth postulate using the other axioms, although unsuccessful, resulted in a new statement that is equivalent to Euclid's Fifth Postulate. This new postulate is now known as Playfair's Axiom in honor of John Playfair, whose commentary, written in 1795, suggested replacing Euclids Fifth Postulate with this axiom.

Playfairs Axiom: Through a point not on a line there is exactly one line parallel to the given line.

We will eventually see that Euclid's Fifth Postulate is independent of the other four Postulates. For more on the history of the development of Axioms for Geometry, see Section 2.1.2 and Appendices A and B in your textbook.

2. Explain how one would go about proving that Euclid's Fifth Postulate is independent.

What Axioms Will We Use?

Since the primary purpose of this course is to prepare teachers to teach high school geometry, we will use the SMSG axiom set, which was developed by the School Mathematics Study Group in 1961 as a suggestion for use with high school geometry courses. Note how the SMSG axioms are a blend of Hilbert's and Birkhoff's axioms. We will begin by introducing several analytic models to illustrate and develop better understanding of the axioms and concepts.

Definition: A **distance** function on a set S is function $d : S \times S \to \mathbb{R}$ such that for all $A, B]inS$

- $d(A, B) \geq 0$
- $d(A, B) = 0$ if and only if $A = B$.
- $d(A, B) = d(B, A)$.

The Geometric Model for Discrete Planes:

Let S be a non-empty set.

- **Points** in this model are elements $s \in S$.
- **Lines** in this model are a specified collection of non-empty subsets of the set S.
- The **distance** function $d : S \to [0, \infty)$ is defined by $d(P, Q) = 0$ if $P = Q$ and $d(P, Q) = 1$ if $P \neq Q$.

3. Verify that the distance function for discrete planes satisfies the three properties of a distance function given on the definition above.

4. Let $S = \{a, b, c, d\}$. Give an example of a discrete plane that contains exactly 8 distinct lines. Be sure to clearly specify your collection of lines. Do the lines in your plane all contain the same number of points? Do any pairs of lines have more than one point in common?