Math 487 - Foundations of Geometry
Day 6 Group Assignment **Name:**______________________________

Analytic Models for Plane Geometry: We will add additional features to these models later, but for today, our focus will be on defining points, lines and distance for several geometric models (as we did for discrete planes last time).

The Riemann Sphere: Points in this model are elements of the unit sphere: $S^2 = \{(x, y, z \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\}$

Lines in this model are the *great circles*, G, of the unit sphere formed by intersecting S^2 and a plane through the origin. That is, there exist $a, b, c \in \mathbb{R}$, not all zero such that $G = \{(x, y, x) \,|\, ax + by + cz = 0\}$.

1. Let P and Q be points on the Riemann Sphere. Explain, in your own words, as clearly as you can, how you would measure the distance between these two points (conceptually).

The Modified Riemann Sphere: Points in this model are formed by taking the points in the Riemann Sphere and identifying pairs of antipodal points. That is, each point in the Modified Riemann Sphere is an equivalence class containing two points of the form $\{(x, y, z), (-x, -y, -z)\}$

Lines in this model are *modified great circles.* This is, we take a great circle G, as defined above, and identify antipodal points, as we did when defining points in our model.

Note: The Modified Riemann Sphere is often visualized as a hemisphere (with only half of the "equator" present).

2. Notice that $P = (-1, 0, 0)$ is a point on the Riemann Sphere. Find P', the antipodal point related to P and graph them in both the Riemann and Modified Riemann sphere. How many great circles contain both of these points?

3. Explain how one might define distance in the Modified Riemann Sphere by adapting the notion of distance used in the Riemann Sphere.

The Cartesian Plane: Points in this model are ordered pairs of the form (x, y) where $x, y \in \mathbb{R}$

Lines in this model are either *vertical lines* of the form $\ell_a = \{(x, y) \,|\, x = a\}$, or non-vertical lines of the form $\ell_{m,b} = \{(x, y) \,|\, y = mx + b\}$.

There are many ways to define a distance in this setting – each leads to a different geometry.

The Euclidean Plane: In this plane, points and lines are taken from the Cartesian Plane, and the distance between two points $P = (x_1, y_1)$, $Q = (x_2, y_2)$ is defined as follows: $d_E(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

The Taxicab Plane: In this plane, points and lines are taken from the Cartesian Plane, and the distance between two points $P = (x_1, y_1)$, $Q = (x_2, y_2)$ is defined as follows: $d_T(P, Q) = |x_2 - x_1| + |y_2 - y_1|$.

The Max - Distance Plane: In this plane, points and lines are taken from the Cartesian Plane, and the distance between two points $P = (x_1, y_1)$, $Q = (x_2, y_2)$ is defined as follows: $d_M(P, Q) = \max\{|x_2 - x_1|, |y_2 - y_1|\}$.

4. Find the distance between the points $P = (-3, 4)$ and $Q = (5, -1)$ in each of these three planes.

5. Draw diagrams illustrating each of the three distances you found above.

The Missing Strip Plane: Points in this model are ordered pairs of the form $M = \{(x, y) \in \mathbb{R}^2 \,|\, x < 0 \text{ or } x \geq 1$.

Lines in this model are modified Cartesian lines. That is, the set of lines is $L = \{\ell \cap M \,|\, \ell$ is a Cartesian line and $\ell \cap M \neq \emptyset$.

The distance between two points $P = (x_1, y_1)$, $Q = (x_2, y_2)$ is defined as follows: If P and Q are on a vertical line, the distance is just the standard Euclidean Distance. Otherwise, $d_{MS}(P, Q) = |g_\ell(x_1, y_1) - g_\ell(x_2, y_2)|$, where ℓ is the Cartesian line containing both P and Q, m is the slope of the line ℓ, and g_ℓ is defined using the standard ruler in the Euclidean Plane (which we will discuss more later – see your textbook for more details), f_ℓ, and

$$g_\ell(x, y) = \begin{cases} f_\ell(x, y) & x < 0 \\ f_\ell(x, y) - \sqrt{1 + m^2} & x \geq 1 \end{cases}$$

6. Draw a representation of the Missing Strip Plane, indicating at least one modified vertical line and at least one modified non-vertical line.

The Poincaré Plane: Points in this model are ordered pairs of the form $H = \{(x, y) \in \mathbb{R}^2 \,|\, y > 0$.

Lines in this model are of two different types:

- vertical rays: ${}_a\ell = \{(x, y) \in H \,|\, x = a\}$.
- Type II lines ("semi-circles" with center point $(c, 0)$): ${}_c\ell_r = \{(x, y) \in H \,||, (x - c)^2 + y^2 = r^2\}$

The distance between two points $P = (x_1, y_1)$, $Q = (x_2, y_2)$ is defined as follows:

$$d_H(P, Q) = \begin{cases} \left|\ln\left(\frac{y_2}{y_1}\right)\right| & \text{if } x_1 = x_2 \\ \left|\ln\left(\frac{\frac{x_2 - c + r}{y_2}}{\frac{x_1 - c + r}{y_1}}\right)\right| & \text{if } P, Q \text{ are on } {}_c\ell_r \end{cases}$$