Math 487 - Foundations of Geometry
Day 7 Group Assignment **Name:**______________________________

Analytic Models and Incidence Axioms:

Recall: In the Missing Strip Plane, the distance between two points $P = (x_1, y_1)$, $Q = (x_2, y_2)$ is defined as follows: If P and Q are on a vertical line, the distance is just the standard Euclidean Distance. Otherwise, $d_{MS}(P,Q) = |g_\ell(x_1, y_1) - g_\ell(x_2, y_2)|$, where ℓ is the Cartesian line containing both P and Q, m is the slope of the line ℓ, and g_ℓ is defined using the standard ruler in the Euclidean Plane, f_ℓ, via: $g_\ell(x,y) = \begin{cases} f_\ell(x,y) & x < 0 \\ f_\ell(x,y) - \sqrt{1+m^2} & x \geq 1 \end{cases}$

Notes: For non-vertical lines, $f_\ell(x,y) = x\sqrt{1+m^2}$, where m is the slope of the line ℓ and for vertical lines $f_\ell(x,y) = y$

In the Poincare Plane, lines are of two different types:

- vertical rays: ${}_a\ell = \{(x,y) \in H \mid x = a\}$.
- Type II lines ("semi-circles" with center point $(c, 0)$): ${}_c\ell_r = \{(x,y) \in H \mid (x-c)^2 + y^2 = r^2\}$

The distance between two points $P = (x_1, y_1)$, $Q = (x_2, y_2)$ is defined as follows: $d_H(P,Q) = \begin{cases} \left|\ln\left(\frac{y_2}{y_1}\right)\right| & \text{if } x_1 = x_2 \\ \left|\ln\left(\frac{\frac{x_2-c+r}{y_2}}{\frac{x_1-c+r}{y_1}}\right)\right| & \text{if } P, Q \text{ are on } {}_c\ell_r \end{cases}$

1. Consider the points $P = (2,1)$, $Q = (2,5)$, $R = (2,9)$, $S = (3,1)$, and $T = (-1,1)$.

   (a) Plot P, Q, R, S, and T as points in the Cartesian Plane. Then find $d_E(T,P)$, $d_E(T,S)$, $d_E(P,Q)$ and $d_E(P,R)$

   (b) Find the $d_{MS}(T,P)$ and $d_{MS}(T,S)$. How do these values compare to those that you found in part (a)? What, if anything, does this tell you about distance in the Missing Strip Plane? Would the difference between these two distances change if we looked at $d(T,Q)$ instead?

   (c) Find the $d_H(P,Q)$ and $d_H(P,R)$. How do these values compare to those that you found in part (a)? What, if anything, does this tell you about distance in the Poincaré Plane?

2. Prove that for two arbitrary points $P = (x_1, y_1)$, $Q = (x_2, y_2)$, the function defined via: $d_E(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ satisfies the definition of a distance function.

Incidence Axioms:

Postulate 1 (Line Uniqueness) Given any two distinct points, there is exactly one line that contains them.

Postulate 5a (Existence of Points) Every plane contains at least three non-collinear points.

Definition: A set of points S is **collinear** if there is a line ℓ such that S is a subset of ℓ. If no such line exists, we say that S is **non-collinear**. If $\{A, B, C\}$ is a collinear set, then we say that the points a, B, and C are **collinear**.

3. Give an example of a collinear set containing three points in the Euclidean Plane.

4. Give an example of a non-collinear set containing three points in the Euclidean Plane.

5. Does the Euclidean Plane satisfy Line Uniqueness? What would you need to show in order to prove this?

6. Give an example of a plane geometry that *does not* satisfy Line Uniqueness. Include a specific example that verifies your claim.

Theorem 2.2 Two distinct lines intersect in at most one point.

7. Prove Theorem 2.2