Math 487 - Foundations of Geometry
Day 9 Group Assignment **Name:**____________________________

Distance and Ruler Axioms:

What is a Ruler? We have all have most likely used an object called a ruler sometime in our lives. What does this device do? Fundamentally, it allows us to measure the distance between two points. But precisely how does this work? We can think of a ruler as a "portable line" with "markings" on it (coordinates of some sort?). We use it by "placing" the ruler so that both points are incident with the line of the ruler. The distance between the points is found by subtracting numbers associated with the markings on the ruler. Let's formalize this a bit using some postulates.

Postulate 2(The Distance Postulate) To every pair of distinct points there corresponds a unique positive number. This number is called the distance between the two points.

Postulate 3(The Ruler Postulate) The points of a line can be placed in correspondence with the real numbers such that:

- To every point of the line there corresponds exactly one real number.
- To every real number there corresponds exactly one point of the line.
- The distance between two distinct points is the absolute value of the difference of the corresponding real numbers.

Postulate 4(The Ruler Placement Postulate) Given two points P and Q of a line, the coordinate system can be chosen in such a way that the coordinate of P is zero and the coordinate of Q is positive.

1. Does every geometry with a distance function (as defined on DGW 5) satisfy Postulate 2? Justify your answer.

The Euclidean Ruler: The standard ruler for a *vertical* line $f : \ell_a \to \mathbb{R}$ is defined by $f(a, y) = y$.

The standard ruler for a *non-vertical* line $f : \ell_{m,b} \to \mathbb{R}$ is defined by $f(x, y) = x\sqrt{1 + m^2}$.

The Taxicab Ruler: The standard ruler for a *vertical* line $f : \ell_a \to \mathbb{R}$ is defined by $f(a, y) = y$.

The standard ruler for a *non-vertical* line $f : \ell_{m,b} \to \mathbb{R}$ is defined by $f(x, y) = (1 + |m|)x$.

The Max Distance Ruler: The standard ruler for a *vertical* line $f : \ell_a \to \mathbb{R}$ is defined by $f(a, y) = y$.

The standard ruler for a *non-vertical* line $f : \ell_{m,b} \to \mathbb{R}$ is defined by $f(x, y) = \begin{cases} x, & \text{if } |m| \leq 1 \\ |m|x & \text{if } |m| > 1 \end{cases}$

2. Let $P = (2, 1)$ and $Q = (5, 5)$ be points in the Cartesian Plane.
   (a) Find coordinates for P and Q using the standard Euclidean Ruler. Verify that the difference corresponds to the distance between them.

(b) Find coordinates for P and Q using the standard Taxicab Ruler. Verify that the difference corresponds to the distance between them.

(c) Find coordinates for P and Q using the standard Max-Distance Ruler. Verify that the difference corresponds to the distance between them.

Definition: A **ruler** or **coordinate system** is a function mapping the points of a line into the real numbers, $f : \ell \to \mathbb{R}$ that satisfies SMSG Postulate 3.

Note: the first and second conditions of the Ruler Postulate imply that f is a one-to-one and onto function. As a reminder, we write the definitions for one-to-one and onto functions.

Definition: A function $f : A \to B$ is **onto** B if for any $b \in B$ there is at least one $a \in A$ such that $f(a) = b$.

Definition: A function $f : A \to B$ is one-to-one if for any $x, y \in A$ with $x \neq y$, then $f(x) \neq f(y)$ [Note that the contrapositive of this definition is often used when writing proofs.]

3. Prove that the standard ruler for *non-vertical* lines in the Euclidean Plane $f : \ell_{m,b} \to \mathbb{R}$, defined by $f(x, y) = x\sqrt{1 + m^2}$, is onto.

4. Prove that the standard ruler for *non-vertical* lines in the Euclidean Plane $f : \ell_{m,b} \to \mathbb{R}$, defined by $f(x, y) = x\sqrt{1 + m^2}$, is one-to-one.

You may spend any remaining time discussing and working on the assigned homework with your group.