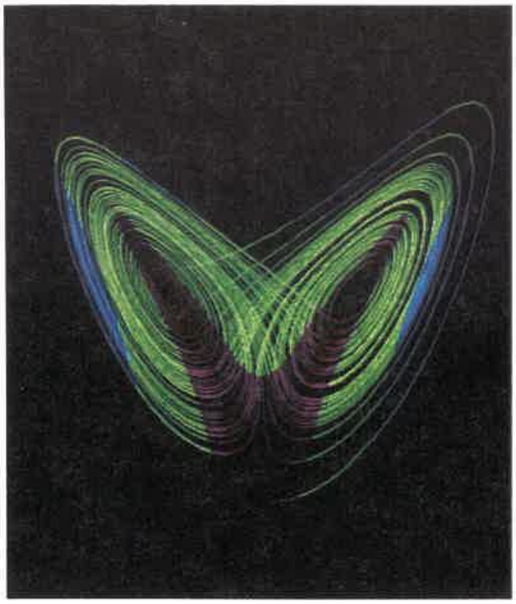


This "strange attractor" represents limit behavior that appeared first in weather models studied by meteorologist E. Lorenz in 1963.

2 LIMITS

Calculus is usually divided into two branches, differential and integral, partly for historical reasons. The subject grew out of efforts in the seventeenth century to solve two important geometric problems: finding tangent lines to curves (differential calculus) and computing areas under curves (integral calculus). However, calculus is a broad subject with no clear boundaries. It includes other topics, such as the theory of infinite series, and it has an extraordinarily wide range of applications. What makes these methods and applications part of calculus is that they all rely on the concept of a limit. We will see throughout the text how limits allow us to make computations and solve problems that cannot be solved using algebra alone.

This chapter introduces the limit concept and sets the stage for our study of the derivative in Chapter 3. The first section, intended as motivation, discusses how limits arise in the study of rates of change and tangent lines.

2.1 Limits, Rates of Change, and Tangent Lines

Rates of change play a role whenever we study the relationship between two changing quantities. Velocity is a familiar example (the rate of change of position with respect to time), but there are many others, such as

- The infection rate of an epidemic (*newly infected individuals per month*)
- Inflation rate (*change in consumer price index per year*)
- Rate of change of atmospheric temperature with respect to altitude

Roughly speaking, if y and x are related quantities, the rate of change should tell us how much y changes in response to a unit change in x. For example, if an automobile travels at a velocity of 80 km/hr, then its position changes by 80 km for each unit change in time (the unit being 1 hour). If the trip lasts only half an hour, its position changes by 40 km, and in general, the change in position is $80t$ km, where t is the change in time (that is, the time elapsed in hours). In other words,

$$\text{Change in position} = \text{velocity} \times \text{change in time}$$

However, this simple formula is not valid or even meaningful if the velocity is not constant. After all, if the automobile is accelerating or decelerating, which velocity would we use in the formula?

The problem of extending this formula to account for changing velocity lies at the heart of calculus. As we will learn, differential calculus uses the limit concept to define *instantaneous velocity*, and integral calculus enables us to compute the change in position in terms of instantaneous velocity. But these ideas are very general. They apply to all rates of change, making calculus an indispensable tool for modeling an amazing range of real-world phenomena.

In this section, we discuss velocity and other rates of change, emphasizing their graphical interpretation in terms of *tangent lines*. Although at this stage, we cannot define precisely what a tangent line is—this will have to wait until Chapter 3—you can think of a tangent line as a line that *skims* a curve at a point, as in Figures 1(A) and (B) but not (C).

2

LIMITS AND CONTINUITY

OVERVIEW Mathematicians of the seventeenth century were keenly interested in the study of motion for objects on or near the earth and the motion of planets and stars. This study involved both the speed of the object and its direction of motion at any instant, and they knew the direction was tangent to the path of motion. The concept of a limit is fundamental to finding the velocity of a moving object and the tangent to a curve. In this chapter we develop the limit, first intuitively and then formally. We use limits to describe the way a function varies. Some functions vary *continuously*; small changes in x produce only small changes in $f(x)$. Other functions can have values that jump, vary erratically, or tend to increase or decrease without bound. The notion of limit gives a precise way to distinguish between these behaviors.

2.1 Rates of Change and Tangents to Curves

Calculus is a tool to help us understand how functional relationships change, such as the position or speed of a moving object as a function of time, or the changing slope of a curve being traversed by a point moving along it. In this section we introduce the ideas of average and instantaneous rates of change, and show that they are closely related to the slope of a curve at a point P on the curve. We give precise developments of these important concepts in the next chapter, but for now we use an informal approach so you will see how they lead naturally to the main idea of the chapter, the *limit*. You will see that limits play a major role in calculus and the study of change.

Average and Instantaneous Speed

HISTORICAL BIOGRAPHY*

Galileo Galilei
(1564–1642)

In the late sixteenth century, Galileo discovered that a solid object dropped from rest (not moving) near the surface of the earth and allowed to fall freely will fall a distance proportional to the square of the time it has been falling. This type of motion is called **free fall**. It assumes negligible air resistance to slow the object down, and that gravity is the only force acting on the falling body. If y denotes the distance fallen in feet after t seconds, then Galileo's law is

$$y = 16t^2,$$

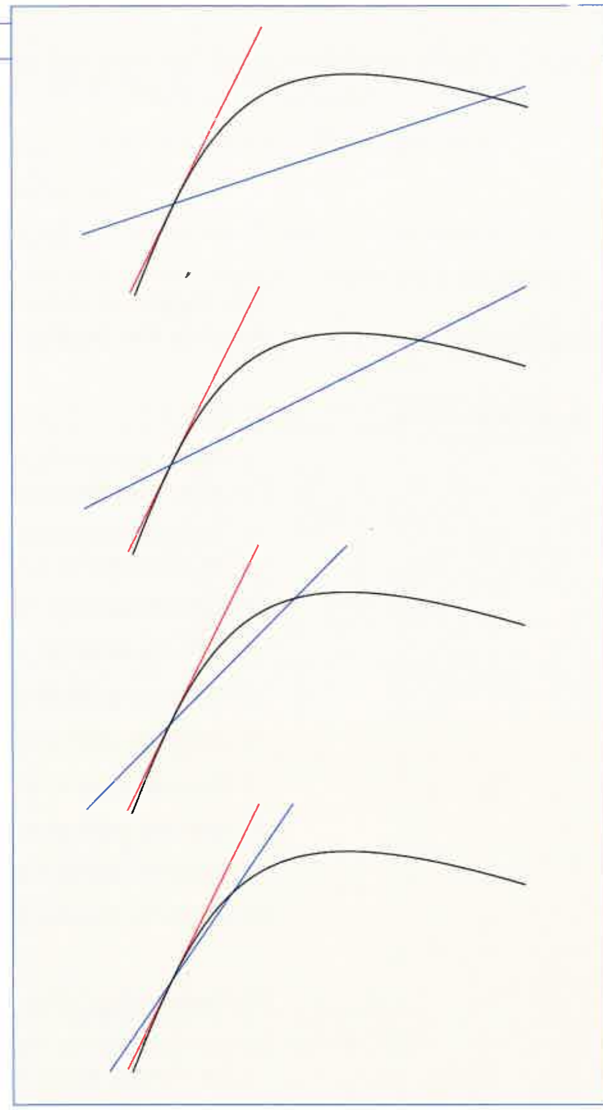
where 16 is the (approximate) constant of proportionality. (If y is measured in meters, the constant is 4.9.)

A moving body's **average speed** during an interval of time is found by dividing the distance covered by the time elapsed. The unit of measure is length per unit time: kilometers per hour, feet (or meters) per second, or whatever is appropriate to the problem at hand.

*To learn more about the historical figures mentioned in the text and the development of many major elements and topics of calculus, visit **www.aw.com/thomas**.

2

LIMITS AND DERIVATIVES



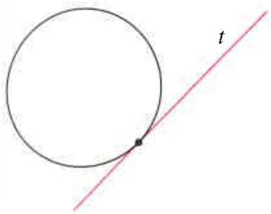
The idea of a limit is illustrated by secant lines approaching a tangent line.

In *A Preview of Calculus* (page 2) we saw how the idea of a limit underlies the various branches of calculus. It is therefore appropriate to begin our study of calculus by investigating limits and their properties. The special type of limit that is used to find tangents and velocities gives rise to the central idea in differential calculus, the derivative.

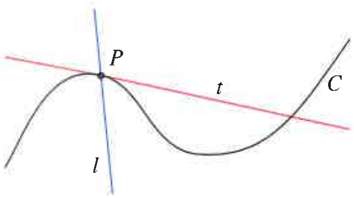
2.1 THE TANGENT AND VELOCITY PROBLEMS

In this section we see how limits arise when we attempt to find the tangent to a curve or the velocity of an object.

THE TANGENT PROBLEM



(a)



(b)

FIGURE 1

The word *tangent* is derived from the Latin word *tangens*, which means "touching." Thus a tangent to a curve is a line that touches the curve. In other words, a tangent line should have the same direction as the curve at the point of contact. How can this idea be made precise?

For a circle we could simply follow Euclid and say that a tangent is a line that intersects the circle once and only once as in Figure 1(a). For more complicated curves this definition is inadequate. Figure l(b) shows two lines l and t passing through a point P on a curve C. The line l intersects C only once, but it certainly does not look like what we think of as a tangent. The line t, on the other hand, looks like a tangent but it intersects C twice.

To be specific, let's look at the problem of trying to find a tangent line t to the parabola $y = x^2$ in the following example.

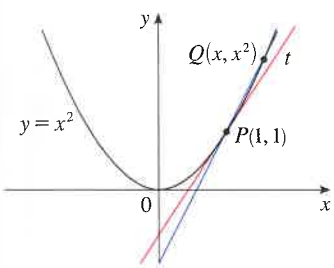


FIGURE 2

x	m_{PQ}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

x	m_{PQ}
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

V EXAMPLE 1 Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

SOLUTION We will be able to find an equation of the tangent line t as soon as we know its slope m. The difficulty is that we know only one point, P, on t, whereas we need two points to compute the slope. But observe that we can compute an approximation to m by choosing a nearby point $Q(x, x^2)$ on the parabola (as in Figure 2) and computing the slope m_{PQ} of the secant line PQ.

We choose $x \neq 1$ so that $Q \neq P$. Then

$$m_{PQ} = \frac{x^2 - 1}{x - 1}$$

For instance, for the point $Q(1.5, 2.25)$ we have

$$m_{PQ} = \frac{2.25 - 1}{1.5 - 1} = \frac{1.25}{0.5} = 2.5$$

The tables in the margin show the values of m_{PQ} for several values of x close to 1. The closer Q is to P, the closer x is to 1 and, it appears from the tables, the closer m_{PQ} is to 2. This suggests that the slope of the tangent line t should be $m = 2$.

We say that the slope of the tangent line is the *limit* of the slopes of the secant lines, and we express this symbolically by writing

$$\lim_{Q \to P} m_{PQ} = m \qquad \text{and} \qquad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

Assuming that the slope of the tangent line is indeed 2, we use the point-slope form of the equation of a line (see Appendix B) to write the equation of the tangent line through $(1, 1)$ as

$$y - 1 = 2(x - 1) \qquad \text{or} \qquad y = 2x - 1$$