

## Part 1: New Content Since the Previous Exam

### Congruences

- Understand the definition of congruence modulo  $n$  and be able to determine whether or not two given integers are congruent modulo  $n$  for some given  $n$ .
- Be able to prove that congruence modulo  $n$  is reflexive, symmetric, and transitive (and thus is an equivalence relation).
- Know the statement of Theorem 6.4.5.
- Be able to prove basic facts about congruence modulo  $n$  such as those outlined in Theorem 6.4.6.
- Understand the definition of a congruence class modulo  $n$  and be able to find the congruence of any integer for some given  $n$ .
- Know the statement of Theorem 6.5.5 and understand that congruence classes modulo  $n$  form a partition of  $\mathbb{Z}$ .
- Understand “canonical representatives” for congruence classes modulo  $m$  and how they are used to define  $\mathbb{Z}_m$ .
- Understand the definition of addition, subtraction, and multiplication in  $\mathbb{Z}_m$ . Be able to perform computations and build operation tables for a given value of  $m$ .
- Know the statement of Theorem 6.5.6 and how it related to the well-definedness of the operations in  $\mathbb{Z}_m$ .

### Relations

- Know the definition of a (binary) relation and how it is represented as a set of ordered pairs.
- Be able to find the domain of a relation.
- Understand that every function is a relation but that not every relation is a function.
- Understand what it means for a relation to be reflexive, symmetric, antisymmetric, or transitive.
- Be able to determine whether a given relation on a set is reflexive, symmetric, antisymmetric, or transitive.
- Be able to combine relations to form new ones using union, intersection, complementation, or difference.
- Given a specific relation be able to represent it using a directed graph.
- Given a directed graph, be able to list the ordered pairs in the relation represented by the directed graph.
- Be able to recognize whether a relation on a set is reflexive, symmetric, antisymmetric, or transitive by looking at its directed graph.

### Equivalence Relations

- Know the definition of an equivalence relation and be able to determine (and prove) whether a given relation is an equivalence relation.
- Understand the definition of the equivalence class of an element with respect to an equivalence relations on a set  $A$  and be able to find the equivalence class of a given element.
- Understand the relationship between equivalence classes and partitions of the underlying set. Be also to tell whether or not a given family of subsets is a partition, and if so, be able find the equivalence relation described by the partition.
- Given a directed graph, be able to determine whether the associated relation is an equivalence relation.

### Partial Orderings

- Know the definition of a partial order and be able to prove whether a given relation is a partial order.
- Understand the definition of comparable and incomparable elements, and be able to determine whether a given pair of elements are comparable or incomparable with respect to a particular partial order.
- Understand definition of a total ordered set and a well ordered set.
- Be able to draw the Hasse diagram for a poset. Also be able to read off the ordered pairs in a partial order from its Hasse diagram.
- Understand the definition of maximal elements, minimal elements, greatest elements, least elements, upper bounds, lower bounds, least upper bounds, greatest lower bounds, and the definition of a lattice. Also be able to find these (when they exist) for elements in a given poset.

## Part 2: Cumulative Content From Previous Exams

### Mathematical Logic

- Understand the definition of a proposition and be able to determine whether or not a given statement is a proposition.
- Understand the definition and truth tables of the logical operators  $\neg$ ,  $\wedge$ ,  $\vee$ ,  $\Rightarrow$ , and  $\Leftrightarrow$
- Be able to build the truth table of any compound proposition.
- Be able to translate back and forth between English statements and symbolic logical propositions.
- Understand the various ways of expressing a conditional statement in English (e.g. necessary, sufficient, only if, whenever, ...). Also know the difference between a conditional statement and its converse, inverse, and contrapositive statement.
- Understand what it means for two propositions to be logically equivalent and be able to prove the logical equivalence of a pair of propositions by building truth tables.
- Understand the what it means for a proposition to be a *tautology*, a *contradiction*, or a *contingency*.
- Be able to prove that a proposition is a tautology.
- Be able to negate propositions using De Morgan's Laws and other equivalences.
- Understand the definition of a predicate involving one or more variables.
- Understand the definition of the universal quantifier, the existential quantifier, and the uniqueness quantifier.
- Be able to determine the truth value of statements involving predicates and/or quantifiers.
- Understand the definition of a *counterexample* and how to use them to show that a statement is false.
- Understand how to find a "useful denial" of a statement involving predicates and quantifiers.
- Understand "hidden quantifiers" and how "mixed quantifiers" can work together to form a statement and the importance of the order of quantification in a statement.
- Be able to determine the truth value of statements involving multiple and "mixed" quantifiers.
- Be able to negate statements involving multiple and "mixed" quantifiers.

### Basic Proof Techniques

- Understand and be able to apply logical argument forms such as *modus ponens*.
- Be able to read a mathematical statement and understand the "Given" and the "Goal" of the statement.
- Be able to write proofs by utilizing: Direct proof, Proof by Contraposition, and Proof by Contradiction.
- Memorize and be able to apply the definitions of odd numbers, even numbers, rational numbers, prime numbers, absolute value, and divisibility.
- Be able to use the Basic Properties of Integers and Real Numbers to prove related properties (you are **not** expected to memorize these).
- Understand and be able to make use of "closure properties" of integers, rational and real numbers.
- Understand and be able to carry out a "for all" proof.
- Understand and be able to carry out an "existence" proof.
- Understand and be able to carry out a "uniqueness" proof.
- Understand and be able to carry out proofs of "biconditional" statements.
- Understand and be able to carry out proofs of "or" statements.
- Understand and be able to utilize Proof by Cases (the Method of Exhaustion).
- Understand how to rule out unnecessary cases (without loss of generality statements).
- Understand the role of counterexamples in disproving statements.
- Understand how to use "backwards reasoning" to help find a proof for a given statement.
- Understand how to form conjectures and how to use examples to investigate a statement.

### Mathematical Induction

- Know the definition of a sequence along with standard sequence notation.
- Understand both explicitly and recursively defined sequences and be able to use them to write out the first few terms in a sequence.
- Understand summation notation and be able to compute summations.
- Understand the principle of mathematical induction and why it is a valid method of proof.
- Be able to prove the base case of a set of statements of the form  $P(n)$  for all  $n > 0$
- Be able to prove theorems using mathematical induction and strong induction.
- Be able to recognize errors in false induction "proofs".

## Sets

- Understand the definitions of sets, elements, subsets, as well as symbolic notation for these terms.
- Understand both roster and set builder notation, and be able to determine whether a given set is well defined.
- Understand the definition of set equality and the definition of the cardinality of a set.
- Understand the definitions of the universal set, the empty set, and the power set of a set.
- Understand how to form the Cartesian Product of two sets.
- Understand the definition of the set operations: union, intersection, complementation, and set difference.
- Be able to determine which elements are in a set resulting from multiple set operations. Also be able to draw a Venn diagram representing such a set.
- Be able to prove statements and properties of sets using containment arguments.
- Understand how to take the union or the intersection of many sets at the same time (including infinite unions and intersections).

## Functions

- Understand the definition of a function, including the definition of the domain, co-domain, and range of a function and be able to apply these definitions to specific functions. Also know how to find the implicit domain of a function.
- Understand and be able to apply the definitions of image and preimage (for both individual elements and sets of elements).
- Understand function equality and be able to determine whether or not a pair of functions are equal.
- Understand the definition of the graph of a function, and be able to draw the graph of a given function.
- Know the definition of a polynomial function, including the degree of a polynomial.
- Know and be able to apply the definition of the identity function on a set.
- Understand the definition of function composition and be able to find composite functions and to prove facts about composite functions.
- Know the definition of one-to-one and onto, and be able to determine whether or not a given function satisfies these definitions.
- Understand the definition of an inverse function, know the properties that a function must satisfy in order to have an inverse, and be able to find the inverse of a function or to determine whether or not a given pair of functions are inverses.