Math 311 - Introduction to Proof and Abstract Mathematics
Group Assignment # 10 **Name:**____________________
Due: In class on Tuesday, February 26th

1. Consider the expression: $\sum_{k=1}^{n}(2k-1)$, where n is a positive integer.

Notice that when $n = 1$ this gives $\sum_{k=1}^{1}(2k-1) = 2(1) - 1 = 2 - 1 = 1$.

When $n = 2$, we have $\sum_{k=1}^{2}(2k-1) = [2(1)-1] + [2(2)-1] = (2-1) + (4-1) = 1 + 3 = 4$.

(a) Compute the value of this expression when $n = 3$, $n = 4$, and $n = 5$.

(b) Make a reasonable conjecture about the value of this expression for an unspecified value of n (your answer should be a formula given in terms of n).

Question: What would we need to do in order to fully prove this conjecture?

Theorem 3.1.1 (The Principle of Mathematical Induction [PMI])
Let $P(n)$ be a statement about the positive integer n, so that n is a free variable in $P(n)$. Suppose the following:

- (PMI 1) The statement $P(1)$ is true.
- (PMI 2) For all positive integers m, if $P(m)$ is true, then $P(m+1)$ is true.

Then, for all positive integers n, the statement $P(n)$ is true.

Notes:

- The statement $P(1)$ in (PMI 1) is called the **Base Case**.
- The statement $P(n)$, the *hypothesis* of the conditional statement given in (PMI 2) is called the **induction hypothesis**. It is the first component of the quantified conditional statement $(\forall m \in \mathbb{N})[P(m) \Rightarrow P(m+1)]$
- In order to see why Theorem 3.1.1 is logically defensible, we can look at it both metaphorically and logically. Metaphorically, we can visualize the infinite chain of statements $P(1)$, $P(2)$, $P(3)$, $\cdots$, $P(n)$, $\cdots$ as steps on a staircase with an infinite number of steps. (PMI 1) tells us that we can step onto the first step ($P(1)$ is true). (PMI 2) tells us that once we have climbed onto the nth step , we can always move up one additional step (if $P(m)$ us true, then $P(m+1)$ is also true). Since we can get onto the staircase and we can always advance one more step, every stair on the staircase is "reachable" ($P(n)$ is true for any n).

  More practically, we can think (PMI) as applying *modus ponens* multiple times. If $P(1)$ is true, and $(\forall m \in \mathbb{N})[P(m) \Rightarrow P(m+1)]$, then, in particular, setting $m = 1$, we have $P(1) \Rightarrow P(2)]$, so, by *modus ponens*, $P(2)$ is true. Since we now know that $P(2)$ is true and we still have $(\forall m \in \mathbb{N})[P(m) \Rightarrow P(m+1)]$, setting $m = 2$, we have $P(2) \Rightarrow P(3)]$, so, again by *modus ponens*, $P(3)$ is true. Since we can continue this process ad infinitum, we can conclude that $P(n)$ is true for any $n \geq 1$.

2. Let's use this principle to prove the statement $\sum_{k=1}^{n}(2k-1) = n^2$ (does this match your previous conjecture?)

   (a) Show that $P(1)$ is true.

   (b) Suppose that $P(m)$ is true for some arbitrary positive integer m. Then $\sum_{k=1}^{m}(2k-1) = m^2$.

   In your own words, explain why $\sum_{k=1}^{m+1}(2k-1) = \sum_{k=1}^{m}(2k-1) + (2(m+1)-1)$.

   (c) Using this, explain why $\sum_{k=1}^{m+1}(2k-1) = m^2 + (2(m+1)-1)$.

   (d) Then $\sum_{k=1}^{m+1}(2k-1) = m^2 + (2m+2) - 1 = m^2 + 2m + 1 = (m+1)^2$.
   Summarize what we just proved in this step in your own words.

   (e) In your own words explain why we can now conclude that $\sum_{k=1}^{n}(2k-1) = n^2$ for all $n \geq 1$.

3. Let's work on another example together. Consider the statement $n^3 + 8n + 9$ is divisible by 3 for all integers $n \geq 1$.

   (a) Prove that $P(1)$ is true. That is, when $n = 1$, show that $n^3 + 8n + 9$ is divisible by 3.

(b) Suppose that $P(m)$ is true for some arbitrary positive integer m. Then $3|m^3+8m+9$, so $\exists k \in \mathbb{Z}$ such that $m^3+8m+9=3k$. Consider $(m+1)^3+8(m+1)+9$. Use algebra to verify that $(m+1)^3+8(m+1)+9 = m^3+3m^2+11m+18$.

(c) From this, notice that $(m+1)^3+8(m+1)+9 = m^3+3m^2+11m+18 = (m^3+8m+9)+3m^2+3m+9 = 3k+3m^2+3m+9 = 3(k+m^2+m+3)$. Explain why this proves that $P(m+1)$ is true, and hence $P(n)$ is true for all $n \geq 1$.

4. Consider the statement $2^n < n!$. Recall that $n! = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 3 \cdot 2 \cdot 1$. For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.

(a) Investigate whether or not this statement is true for small positive values of n ($n = 1, 2, 3, 4, ...$).

Suppose that $P(m)$ is true for some reasonable large integer m. That is, suppose $2^m < m!$. Notice that $(m+1)! = m! \cdot (m+1)$. Then, using order properties of integers, $2^m \cdot (m+1) < m! \cdot (m+1)$, so $2^m \cdot (m+1) < (m+1)!$.

(b) Prove that $2^{m+1} = 2^m \cdot 2 < 2^m \cdot (m+1)$.

(c) From this, we may, using the transitivity of $<$, we may conclude that $2^{m+1} < (m+1)!$. Combining this with your work and explorations above, for which n values is the statement $2^n < n!$ true?

5. The remainder of this handout consists of general presentation problems that do not need to be handed in for grading but that you can present on the board during class this week.

   (a) Use the Principle of Mathematical Induction to prove that for all $n \geq 1$, $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

   (b) Use the Principle of Mathematical Induction to prove that for all $n \geq 1$, $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$

   (c) Use the Principle of Mathematical Induction to prove that for all $n \geq 1$, $4^n - 1$ is divisible by 3.

   (d) Use the Principle of Mathematical Induction to prove that for all $n \geq 5$, $n^2 < 2^n$.

   (e) Prove that $m^2 = n^2$ if and only if $m = n$ or $m = -n$

   (f) Prove that the number $\sqrt[3]{2}$ is irrational.

   (g) Prove or disprove: Every non-negative integer can be written as the sum of at most 3 perfect squares.

   (h) Prove or disprove: Let a, b, c and d be integers. If $a|b$ and $c|d$, then $ac|bd$.

   (i) Prove or disprove: If a and b are positive real numbers, then $a + b \geq 2\sqrt{ab}$.

   (j) Prove or disprove: If a does not divide bc, then a does not divide b.

   (k) Formulate a conjecture about the decimal digits that appear as the final digit of the fourth power of an integer. Prove your conjecture using proof by cases.