Math 311 - Introduction to Proof and Abstract Mathematics
Group Assignment # 14 **Name:**____________________
Due: At the end of class on Thursday, March 21st

Arbitrary Unions and Intersections:

1. For each of the following collections of subsets of $\mathbb{R}$, find $\bigcup_{i=1}^{n} A_i$ and $\bigcap_{i=1}^{n} A_i$ for the given n-value.

   (a) $A_i = \{0, 1, 2, 3, \cdots i\}$ given that $n = 5$

   (b) $A_i = [0, i]$ given that $n = 5$

   (c) $A_i = \{0, 1, 2, 3, \cdots i\}$ given that $n = N$

   (d) $A_i = [0, i]$ given that $n = N$

Theorem 4.3.2 Let $n \in \mathbb{Z}^+$. then for all sets $A, B_1, B_2, \cdots, B_n$,

- $A \cup (B_1 \cap B_2 \cap \cdots \cap B_n) = (A \cup B_1) \cap (A \cup B_2) \cap \cdots \cap (A \cup B_n)$.
- $A \cap (B_1 \cup B_2 \cup \cdots \cup B_n) = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n)$.
- $\overline{B_1 \cup B_2 \cup \cdots \cup B_n} = \overline{B_1} \cap \overline{B_2} \cap \cdots \cap \overline{B_n}$.
- $\overline{B_1 \cap B_2 \cap \cdots \cap B_n} = \overline{B_1} \cup \overline{B_2} \cup \cdots \cup \overline{B_n}$.

Proof: We will prove the first part (using proof by induction). The second part is proved in your textbook. The remaining parts are presentation problems.

Base Case: For $n = 1$, we see that $A \cup B_1 = A \cup B_1$, so the statement is true when $n = 1$.

Inductive Step: Suppose that the statement holds for all collections of sets of size m for some $m \geq 1$. That is, suppose that for any collection of sets $C, D_1, D_2, \cdots D_m$, we have $C \cup (D_1 \cap D_2 \cap \cdots \cap D_n) = (C \cup D_1) \cap (C \cup D_2) \cap \cdots \cap (C \cup D_n)$.

Given a collection of sets $A, B_1, B_2, \cdots, B_m, B_{m+1}$, consider $A \cup (B_1 \cap B_2 \cap \cdots \cap B_m \cap B_{m+1})$. Notice that $A \cup (B_1 \cap B_2 \cap \cdots \cap B_m \cap B_{m+1}) = A \cup ((B_1 \cap B_2 \cap \cdots \cap B_m) \cap B_{m+1}) = A \cup (B_1 \cap B_2 \cap \cdots \cap B_m) \cap (A \cup B_{m+1})$ [by Theorem 4.2.6].

By the induction hypothesis, with $C = A$ and $D_i = B_i$, we have $A \cup (B_1 \cap B_2 \cap \cdots \cap B_m) = (A \cup B_1) \cap (A \cup B_2) \cap \cdots \cap (A \cup B_n)$. Hence $A \cup (B_1 \cap B_2 \cap \cdots \cap B_m \cap B_{m+1}) = ((A \cup B_1) \cap (A \cup B_2) \cap \cdots \cap (A \cup B_n)) \cap (A \cup B_{m+1})$, which verifies the $m + 1$ case.

Therefore, $A \cup (B_1 \cap B_2 \cap \cdots \cap B_n) = (A \cup B_1) \cap (A \cup B_2) \cap \cdots \cap (A \cup B_n)$. $\square$.

The next definition extends thus idea to an infinite collection of sets (a collection indexed – that is, put in 1-1 correspondence through labeling – by the positive integers.

Definition 4.3.3: Given sets A_i, $i \in \mathbb{Z}^+$, with underlying universal set $\mathcal{U}$,

$$\bigcup_{i=1}^{\infty} A_i = \{x \in \mathcal{U} \mid \text{ there exists } i \in \mathbb{Z}^+ \text{ such that } x \in A_i\}.$$

$$\bigcap_{i=1}^{\infty} A_i = \{x \in \mathcal{U} \mid \text{ for all } i \in \mathbb{Z}^+,\ x \in A_i\}.$$

2. For each of the following indexed collections of subsets of $\mathbb{R}$, A_i for $i \in \mathbb{Z}^+$, find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$.

(a) $A_i = \{0, 1, 2, 3, \cdots i\}$

(b) $A_i = [0, i)$

(c) $A_i = [0, \frac{1}{i}]$

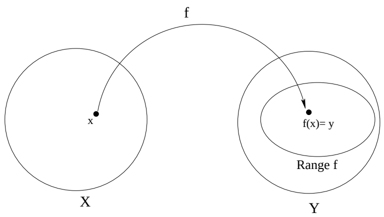
Definition 4.3.6: Let I be a non-empty set and let Given sets $\{A_i, | i \in I\}$ be an indexed family of sets, relative to some universal set $\mathcal{U}$. Then

- For each $j \in I$, $\bigcap_{i=I} A_i \subseteq A_j$.
- For each $j \in I$, $A_j \subseteq \bigcup_{i=I} A_i$.
- $B \cup \bigcap_{i=I} A_i = \bigcap_{i=I}(B \cup A_i)$
- $B \cap \bigcup_{i=I} A_i = \bigcup_{i=I}(B \cap A_i)$
- $\overline{\bigcup_{i=I} A_i} = \bigcap_{i=I} \overline{A_i}$
- $\overline{\bigcap_{i=I} A_i} = \bigcup_{i=I} \overline{A_i}$

3. Explain, in your own words, why the first two parts of the Theorem above make sense.

Functions:

Definition 5.1.1: Let X and Y be non-empty sets. A **function** f from the set X to the set Y, denoted by $f : X \to Y$ or by $f(x)$ is a correspondence that assigns to each element $x \in X$ a unique element $y \in Y$.



We say that f **maps** x to y (written $y = f(x)$) when y is the unique element of Y that is assigned by f to an element x in X. We call y the **image of** x **under** f, and we call x a **preimage of** y **under** f. We say that the function f maps the set X into the set Y.

Continuing to use the notation defined above, we say that the set X is the **domain** of the function f and the set Y is the **codomain** of f.

Definition 5.1.4 Let X and Y be sets and let $f : X \to Y$. The **range** of f (also called the image of X under f) is the set $\{y \in Y \mid (\exists x \in X[y = f(x)]\} = \{f(x) \mid x \in X\}$. We will often denote this as either $ran\, f$ or $im\, f$.

Definition 5.1.5 Let X and Y be sets and let $f : X \to Y$. The **graph of** f is the set $G_f = \{(x, y) \in X \times Y \mid y = f(x)\} = \{(x, f(x)) \mid x \in X\}$.

4. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by the rule $f(n) = 0$ if n is even and $f(n) = 1$ if n is odd.

   (a) Explain, in your own words, why f, as defined above is a function.

   (b) Find the following:

      i. $f(12)$

      ii. $f(27)$

      iii. the image of 0 under f.

      iv. The preimage of 0 under f

      v. The preimage of 1 under f

      vi. The preimage of 2 under f.

      vii. The domain of f

      viii. The range of f

5. Let $f(x) = \sin x$ and $g(x) = \cos\left(x - \frac{\pi}{2}\right)$ be functions from $\mathbb{R} \to \mathbb{R}$.

   (a) Find the domain and range of $f(x)$.

   (b) Find the domain and range of $g(x)$.

   (c) Draw the graph of $f(x)$.

   (d) Draw the graph of $g(x)$.

6. **Presentation Problems:** Let A, B, and C be arbitrary sets.

   (a) Prove Proposition 4.2.12

   (b) Prove $A - (A - B) = A \cap B$

   (c) Prove $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

   (d) Prove $A \times (B \cap C) = (A \times B) \cap (A \times C)$

   (e) Prove or Disprove: If $A \subseteq B \cup C$ then $A \subseteq B$ or $A \subseteq C$.

   (f) Prove or Disprove: If $A \subseteq B \cap C$ then $A \subseteq B$ and $A \subseteq C$.

   (g) Prove or Disprove: If $A - C \subseteq B - C$ then $A \subseteq B$.

   (h) Prove or Disprove: $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$.

   (i) Prove or Disprove: $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

   (j) Express the set $\{1\}$ as the intersection of a collection of distinct, non-empty intervals in $\mathbb{R}$ indexed by $\mathbb{Z}^+$.