Math 311 - Introduction to Proof and Abstract Mathematics
Group Assignment # 15 **Name:**______________________________
Due: At the end of class on Tuesday, March 26th

More on Functions:

Definition 5.1.7: Let A, B, C, and D be sets. Let $f : A \to B$ and $g : C \to D$ be functions. Then $f = g$ if:

- $A = C$ and $B = D$
- For all $x \in A$, $f(x) = g(x)$.

Intuitively speaking, this definition tells us that a function is determined by its underlying correspondence, not its specific formula or rule. Another way to think of this is that a function is determined by the set of points that occur on its graph.

1. Give a specific example of two functions that are defined by different rules (or formulas) but that are equal as functions.

2. Consider the functions $f(x) = x$ and $g(x) = \sqrt{x^2}$. Find:

   (a) A domain for which these functions are equal.

   (b) A domain for which these functions are **not** equal.

Definition 5.1.9 Let X be a set. The **identity function on** X is the function $I_X : X \to X$ defined by, for all $x \in X$, $I_X(x) = x$.

3. Let $f(x) = x\cos(2\pi x)$. Prove that $f(x)$ is the identity function when $X = \mathbb{Z}$ but not when $X = \mathbb{R}$.

Definition 5.1.10 Let $n \in \mathbb{Z}$ with $n \geq 0$, and let $a_0, a_1, \cdots, a_n \in \mathbb{R}$ such that $a_n \neq 0$. The function $p : \mathbb{R} \to \mathbb{R}$ is a **polynomial of degree** n **with real coefficients** $a_0, a_1, \cdots, a_n$ if for all $x \in \mathbb{R}$, $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$.

The **zero polynomial** is the function $q : \mathbb{R} \to \mathbb{R}$ such that $q(x) = 0$ for all $x \in \mathbb{R}$. The degree of this polynomial is undefined.

4. True or False: $p(x) = 1$ is a polynomial (briefly justify your answer).

5. True or False: $p(x) = 1$ is the identity function on $\mathbb{R}$ (briefly justify your answer).

6. True or False: Every polynomial has domain $\mathbb{R}$ (briefly justify your answer).

7. True or False: Every polynomial has image $\mathbb{R}$ (briefly justify your answer).

8. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \dfrac{x+4}{x^2-9}$. For what values of x is $f(x)$ defined?
   Note: we often call this the *implicit* or *natural domain* of f.

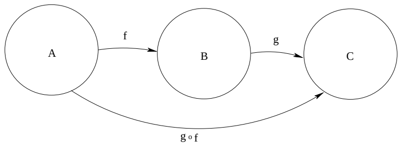
9. Let $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be defined by $g(m,n) = mn$.
   (a) Find the image of the function g.

   (b) Find the preimage of 0 under g.

Function Composition:

Definition 5.2.1: Let A, B, C, and D be sets. Let $f : A \to B$ and $g : C \to D$ with $ran\, f \subset C$. The **composite (composition) of** f **and** g is the function $g \circ f : A \to D$ defined by, for all $x \in A$, $(g \circ f)(x) = g(f(x))$.

Note: When $f : A \to B$ and $g : B \to C$, then we the following diagram illustrates the composite function $g \circ f : A \to C$.



10. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x - 1$ and $g(x) = x^2$ for all $x \in \mathbb{R}$.

    (a) Find $(g \circ f)(4)$

    (b) Find $(f \circ g)(4)$

    (c) Find $(g \circ f)(x)$

    (d) Find $(f \circ g)(x)$

    (e) Based on this example, what do you conclude about how $(g \circ f)(x)$ and $(f \circ g)(x)$ compare in general?

11. Let $f(x) = \begin{cases} x - 2 & \text{if } x \geq 1 \\ x + 2 & \text{if } x < 1 \end{cases}$ and let $g(x) = \begin{cases} 2x & \text{if } x > 4 \\ \frac{1}{2}x & \text{if } x \leq 4 \end{cases}$

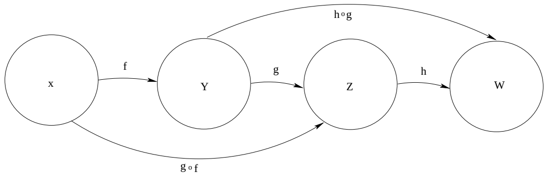
    (a) Find $(g \circ f)(1)$

    (b) Find $(f \circ g)(4)$

    (c) Find $(g \circ f)(x)$

Proposition 5.2.5: Let X, Y, Z, and W be sets. Let $f : X \to Y$, $g : Y \to Z$, and $h : Z \to W$. Then:

- $(h \circ g) \circ f = h \circ (g \circ f)$; that is, function composition is associative.
- $f \circ I_X = f = I_Y \circ f$.



Proof: Let $f : X \to Y$, $g : Y \to Z$, and $h : Z \to W$. To prove the first part, we must demonstrate that $(h \circ g) \circ f = h \circ (g \circ f)$. Note that both $(h \circ g) \circ f$ and $h \circ (g \circ f)$ define functions from X into W. Let $x \in X$. Suppose $y = f(x)$. Then $(h \circ g) \circ f)(x) = (h \circ g)(f(x)) = (h \circ g)(y) = h(g(y)) = h(g(f(x)))$. Similarly, $h \circ (g \circ f)(x) = h((g \circ f)(x))$. But $(g \circ f)(x) = g(f(x)) = g(y)$, so $h((g \circ f)(x)) = h(g(y)) = h(g(f(x)))$. Hence, for any $x \in X$, $(h \circ g) \circ f)(x) = h \circ (g \circ f)(x)$. Hence $(h \circ g) \circ f = h \circ (g \circ f)$.

Let $x \in X$. Recall that by definition, for any $x \in X$, $I_X(x) = x$. Therefore, $(f \circ I_X)(x) = f(I_X(x)) = f(x)$. Hence $f \circ I_X = f$. The proof that $f = I_Y \circ f$ is a similar proof that is left to you. □.

12. Prove that $f = I_Y \circ f$.

13. Find a distinct pair of functions, $f, g : \mathbb{R} \to \mathbb{R}$ such that $f \circ g = g \circ f = I_{\mathbb{R}}$.

Definition 5.3.1 Let X and Y be sets and let $f : X \to Y$.

- The function f is **one-to-one** if $(\forall x_1, x_2 \in X)[x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)]$, or, equivalently, $(\forall x_1, x_2 \in X)[f(x_1) = f(x_2) \Rightarrow x_1 = x_2]$.

  When f is one-to-one, we say that f is **injective**, or that f is **an injection**.

- The function f is **onto** if $(\forall y \in Y)[\exists x \in X \,|\, y = f(x)]$.
  When f is onto, we say that f is **surjective**, or that f is **a surjection**. Note that $f : X \to Y$ is onto iff $ran\, f = Y$.
- The function f is **bijective**, or is a bijection (or a 1-1 **correspondence**), if f is both an injection and a surjection. That is, f is both 1-1 and onto.

Example: Let $f(x) = 3x - 2$. We will prove that $f(x)$ is a bijection from $\mathbb{R}$ to $\mathbb{R}$.

First, to see that $f(x)$ is 1-1, suppose that $f(a) = f(b)$. Then $3a - 2 = 3b - 2$. Adding 2 to both sides gives $3a = 3b$. Dividing both sides by 3 gives $a = b$. Hence f is 1-1.

To see that f is onto, let $y \in \mathbb{R}$. Let $x = \frac{y+2}{3}$ (note that $x \in \mathbb{R}$). Then $f(x) = f\left(\frac{y+2}{3}\right) = 3 \cdot \left(\frac{y+2}{3}\right) - 2 = (y + 2) - 2 = y$. This shows that $y \in im\, f$. Since y was arbitrary, this demonstrates that f is onto.

Since f is both 1-1 and onto, then f is a bijection. □.