Math 311 - Introduction to Proof and Abstract Mathematics
Group Assignment # 16 **Name:**____
Due: At the end of class on Thursday, March 28th

One-to-One, Onto, and Inverse Functions:

Recall: Definition 5.3.1 Let X and Y be sets and let $f : X \to Y$.

- The function f is **one-to-one** if $(\forall x_1, x_2 \in X)[x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)]$, or, equivalently, $(\forall x_1, x_2 \in X)[f(x_1) = f(x_2) \Rightarrow x_1 = x_2]$.

  When f is one-to-one, we say that f is **injective**, or that f is **an injection**.

- The function f is **onto** if $(\forall y \in Y)[\exists x \in X \mid y = f(x)]$.

  When f is onto, we say that f is **surjective**, or that f is **a surjection**. Note that $f : X \to Y$ is onto iff $ran\, f = Y$.

- The function f is **bijective**, or is a bijection (or a 1-1 **correspondence**), if f is both an injection and a surjection. That is, f is both 1-1 and onto.

Theorem 5.3.6 Let $f : X \to Y$ and let $R = ran\, f$ be the range of f. Then the function $g : X \to R$ defined by $g(x) = f(x)$ for all $x \in X$ is onto.

1. For each of the following functions, determine (with proof) whether or not the function is 1-1, onto, both, or neither. For those that are not onto, find the range of the function.

   (a) $f(x) = x^2$ where $f : \mathbb{R} \to \mathbb{R}$.

   (b) $g(n) = 2n$ where $g : \mathbb{N} \to \mathbb{N}$.

   (c) $h(x) = 2x$ where $h : \mathbb{R} \to \mathbb{R}$.

(d) $p(x, y) = x + y$ where $p : \mathbb{R}^2 \to \mathbb{R}$.

Invertible Functions:

Theorem 5.3.6: Let $f : X \to Y$, and let $ran\, f$ be the range of f. Then the function $g : X \to ran\, f$ defined by $g(x) = f(x)$ for all $x \in X$ is onto.

2. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sqrt{x} - 4$. Find the image of the function f, and explain why the associated function $g : \mathbb{R} \to im\, f$ given by $g(x) = f(x)$ for all $x \in X$ is onto.

3. Prove Theorem 5.3.6

Theorem 5.3.8: Let $f : X \to Y$ be onto. Then there exists a subset $X_0 \subseteq X$ such that restricting the domain of f to X_0 yields a 1-1 and onto function; i.e., the function $g : X_0 \to Y$ defined by $g(x) = f(x)$ for all $x \in X_0$ is a bijection.

Notes:

• The proof of Theorem 5.3.8 requires the Axiom of Choice (a controversial axiom which you can read about in section 4.4 of your textbook, or other places). The main requirement is that for each $y \in Y$, one must be able to select a specific $x \in X$ such that $f(x) = y$. This is straightforward when Y is a finite set, but can be problematic when Y is infinite.

• In most of the examples we have looked at so far, we have shown a function is onto by using algebra to produce an element x in the domain such that $f(x) = y$. In practice, using algebra to find such x-values precisely can be challenging. Results like the Intermediate Value Theorem are helpful as they allow us to prove that an input leading to a given value exists without finding its exact value.

• Results like these are said to **non-constructive** existential proofs (in contrast to what are called **constructive** existential proofs) since they show something exists, but do not actually find the object under consideration. Constructive vs. Non-Constructive proof is another area of controversy in mathematics.

4. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin x$. Find $im\, f$. Then find a set X_0 such that the function $g : X_0 \to im\, f$ defined by $g(x) = f(x)$ for all $x \in X_0$ is a bijection.

Theorem 5.3.10 Let X, Y, and Z be sets. Let $f : X \to Y$ and $g : Y \to Z$ be functions.

- If f and g are both 1-1, then $g \circ f$ is 1-1.
- If f and g are both onto, then $g \circ f$ is onto.
- If f and g are both bijections, then $g \circ f$ is a bijection.
- If $g \circ f$ is 1-1, then f is 1-1, but g need not be.
- If $g \circ f$ is onto, then g is onto, but f need not be.

Proof: If f and g are both 1-1, then $g \circ f$ is 1-1.

Suppose $f : X \to Y$ and $g : Y \to Z$ are 1-1 functions. Let $a, b \in X$ and suppose $(g \circ f)(a) = (g \circ f)(b)$. Then $g(f(a)) = g(f(b))$. Since g is 1-1, we must have $f(a) = f(b)$. Then, since f is 1-1, we must have $a = b$. Hence $g \circ f$ is 1-1. $\square$.

5. Prove that If f and g are both onto, then $g \circ f$ is onto.

6. Give an example of functions f and g such that $g \circ f$ is 1-1 and f is 1-1, but g is not 1-1.

Definition 5.4.1 Let X and Y be sets and let $f : X \to Y$. We say that f is **invertible** if there exists a function $g : Y \to X$ such that for all $x \in X$ and for all $y \in Y$, $y = f(x)$ if and only if $x = g(y)$.

When this definition is satisfied, we say that the function g is an **inverse function** of f.

Proposition 5.4.3 Let X and Y be sets, and let $f : X \to Y$ and let $g : Y \to X$. Then f is invertible and g is an inverse function of f if and only if $g \circ f = I_X$ and $f \circ g = I_Y$.

Proof: Let $f : X \to Y$ and $g : Y \to X$.

" $\Rightarrow$". Assume that f is invertible and that g is an inverse function of f. Then, by definition, $y = f(x) \Leftrightarrow x = g(y)$. Consider $g \circ f$. Notice that $g \circ f : X \to X$. Let $x \in X$ and suppose $y = f(x)$. Then $(g \circ f)(x) = g(f(x)) = g(y) = x = I_X(x)$. Hence $g \circ f = I_X$.

Similarly, $f \circ g : Y \to Y$. Let $y \in Y$ and suppose $x = g(y)$. Then $(f \circ g)(y) = f(g(y)) = f(x) = y = I_Y(y)$. Hence $f \circ g = I_Y$.

" $\Leftarrow$". Assume that f and g satisfy $g \circ f = I_X$ and $f \circ g = I_Y$.

Let $x \in X$ and suppose $y = f(x)$. Then $g(y) = g(f(x)) = (g \circ f)(x) = I_X(x) = x$. Hence $g(y) = x$.

Similarly, let $y \in Y$ and suppose $x = g(y)$. Then $f(x) = f(g(y)) = (f \circ g)(y) = I_Y(y) = y$. Hence $f(x) = y$.

Thus f is invertible and g is an inverse function of f. $\square$.

Corollary 5.4.4: Let X and Y be sets and let $f : X \to Y$. Then f is invertible if and only if there is a function $g : Y \to X$ such that $g \circ f = I_X$ and $f \circ g = I_Y$.

7. Let $f(x) = 2x - 3$ and $g(x) = \frac{x+3}{2}$ be functions from $\mathbb{R} \to \mathbb{R}$. Prove that f and g are inverse functions.

Theorem 5.4.7 Let X, Y be sets and let $f : X \to Y$.

- Then f is invertible if and only if f is a bijection.
- If f is invertible, then its inverse function is unique.

Proof: See p. 117 in your textbook.