Math 311 - Introduction to Proof and Abstract Mathematics
Group Assignment # 17 **Name:**____
Due: At the end of class on Thursday, March 28th

Recall: Definition 5.4.1 Let X and Y be sets and let $f : X \to Y$. We say that f is **invertible** if there exists a function $g : Y \to X$ such that for all $x \in X$ and for all $y \in Y$, $y = f(x)$ if and only if $x = g(y)$.

When this definition is satisfied, we say that the function g is an **inverse function** of f.

Notation 5.4.8: When $f : X \to Y$ is invertible, the unique inverse function is denoted by f^{-1}, and $f^{-1} : Y \to X$.

Corollary 5.4.9 Let X and Y be sets and let $f : X \to Y$. If f is a bijection, then $f^{-1} : Y \to X$ is a bijection.

Corollary 5.4.10 Let X and Y be sets and assume $f : X \to Y$ is 1-1. Then the function $g : X \to ran\, f$ defined by, for all $x \in X$, $g(x) = f(x)$ is invertible.

Corollary 5.4.11 Let X and Y be sets and let $f : X \to Y$ and $g : Y \to X$. If $g \circ f = I_X$ and $f \circ g = I_Y$, then $g = f^{-1}$ and $f = g^{-1}$.

1. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f((x, y)) = (2y, 3x)$. Determine whether or not f is invertible. If it is invertible, find f^{-1}, its inverse function.

Definition 5.5.1 Let $f : X \to Y$ and $A \subseteq X$. The **image of** A **under** f is the set $\{y \in Y \,|\, (\exists x \in A)[y = f(x)]\} = \{f(x) \,|\, x \in A\}$. We sometimes denote the image set as $f[A]$.

Definition 5.5.2 Let $f : X \to Y$ and $B \subseteq Y$. The **inverse image of** B **under** f (also called the **preimage** of B under f) is the set $\{x \in X \,|\, f(x) \in B\}$. We sometimes denote the preimage of a set as $f^{-1}[B]$ (even in situations where f is not invertible).

2. Let $f : \mathbb{Z} \to \mathbb{Z}$ be given by $f(k) = k + 11$. Let E be the set of even integers and let F be the set of odd integers. Let $A = \{k \in \mathbb{Z} \,|k \geq 0\}$. and let $B = \{k \in \mathbb{Z} \,|k < 0\}$

(a) Find $f[E]$.

(b) Find $f[A]$.

(c) Find $f^{-1}[F]$

(d) Find $f^{-1}[B]$

Note: We will spend the remainder of the day on Presentation Problems.

(a) Give a **proof** or a **specific counterexample** for the following: If ℓ divides m and m divides n, then ℓ divides n.

(b) Use the Principle of Mathematical Induction to prove that for all $n \geq 1$, $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$

(c) Use the Principle of Mathematical Induction to prove that for all $n \geq 5$, $n^2 < 2^n$.

(d) Use mathematical induction to prove that $x^2 - 1$ is divisible by 8 whenever x is a *positive odd integer.*

(e) Prove or disprove: Let a, b, c and d be integers. If $a|b$ and $c|d$, then $ac|bd$.

(f) Prove that $x^2 + y^2 = 11$ has no integer solutions.

(g) Prove or disprove: If a does not divide bc, then a does not divide b.

(h) Formulate a conjecture about the decimal digits that appear as the final digit of the fourth power of an integer. Prove your conjecture using proof by cases.

(i) Suppose that a fast food restaurant sells chicken nuggets in packs of 4, 7, or 9. What is the largest number of chicken nuggets that you **cannot** buy exactly (Fully justify your answer).

(j) Suppose that a different restaurant sells chicken nuggets in packs of 4 or 15. What is the largest number of chicken nuggets that you **cannot** buy exactly (Fully justify your answer).

(k) Prove Proposition 4.2.12

(l) Prove $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

(m) Prove $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(n) Prove or Disprove: If $A - C \subseteq B - C$ then $A \subseteq B$.

(o) Prove or Disprove: $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$.

(p) Prove or Disprove: $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

(q) Express the set $\{1\}$ as the intersection of a collection of distinct, non-empty intervals in $\mathbb{R}$ indexed by $\mathbb{Z}^+$.

(r) Use the properties of Real Numbers to prove that given $a, b \in \mathbb{R}$, $ab = 0$ if and only if $a = 0$ or $b = 0$.

(s) Let $\mathcal{P}$ be the set of all polynomial functions. Let $D : \mathcal{P} \to \mathcal{P}$ be the function given by $D(p(x)) = p'(x)$. That, the function that maps each function to its derivative. Each of the following is a separate presentation problem:

   i. Find $im\, D$.

   ii. Let $\mathcal{L} = \{p(x) \in \mathcal{P} \,|\, p(x) = ax + b$. Find the preimage of $\mathcal{L}$ under D.

   iii. If D 1-1? Justify your answer.

(t) Let $h(x) = \sqrt{x^2+1} - 5$. Find three different ways to express $h(x)$ as the composition of two functions (that is, find three distinct pairs of functions f and g so that $h = g \circ f$.

(u) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f((x, y)) = x + y$. Show that f is not 1-1, but that f is onto.

(v) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f((x, y)) = x + y$. Find a set $A_0 \subset \mathbb{R}^2$ so that $f : A_0 \to \mathbb{R}$ is a bijection.

(w) Prove Corollary 5.4.10

(x) This problem involves a famous set-theoretic concept called Russell's Paradox. According to Russell's definitions, a set A is called **normal** if A is *not* an element of itself. Similarly, a set is **abnormal** if it *is* an element of itself.

i. Give an example of a set that is normal.
ii. Give an example of a set that is abnormal.
iii. Let $\mathcal{N} = \{A \,|\, A \text{ is a set that is normal }\}$ and let $\mathcal{A} = \{A \,|\, A \text{ is a set that is abnormal }\}$.
Is $\mathcal{N}$ a normal set or an abnormal set? How about $\mathcal{A}$? Explain how this leads to a paradox and comment on what caused things to go wrong.